AN ITERATIVE FORMULA FOR SPECIAL CHEBYSHEV APPROXIMATIONS

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The first part of the paper gives two theorems for characterization of best Chebyshev approximations (the basic idea of the Theorem 1.1 can be found in [2] and [3]). In the second part we shall give a program of the iteration of the Theorem 1.2.

1. On the characteristic properties of the best approximation

The fixed symbols are as follows: $[a, b]\in\mathbf{R}^1$, a closed and bounded interval; $f(x)$, a continuous function on $[a, b]$; $g_1(x), g_2(x), \ldots, g_n(x)$, a Chebyshev system of the continuous functions on $[a, b]$, i.e.

$$\begin{vmatrix} g_1(x_1) & \ldots & g_1(x_n) \\ \vdots & & \vdots \\ g_n(x_1) & \ldots & g_n(x_n) \end{vmatrix} > 0$$

if $a\le x_1<x_2<\ldots<x_n\le b$; $A_1^*, \ldots, A_n^*$, the coefficients of the best Chebyshev approximation, i.e.

$$\max_{x\in[a,b]} |A_1^*g_1(x)+\ldots+A_n^*g_n(x)-f(x)| < \max_{x\in[a,b]} |A_1g_1(x)+\ldots+A_ng_n(x)-f(x)|,$$

where $A^*\equiv\{A_1^*, \ldots, A_n^*\}\in\mathbf{R}^n$, $A\equiv\{A_1, \ldots, A_n\}\in\mathbf{R}^n$ and $A^*\ne A$; $x_1^*, x_2^*, \ldots, x_{n+1}^*$, a set of the extremal points, i.e.

$$A_1^*g_1(x_i^*)+\ldots+A_n^*g_n(x_i^*)-f(x_i^*) = -\left(A_1^*g_1(x_{i+1}^*)+\ldots+A_n^*g_n(x_{i+1}^*)-f(x_{i+1}^*)\right)$$

and

$$|A_1^*g_1(x_i^*)+\ldots+A_n^*g_n(x_i^*)-f(x_i^*)| = \max_{x\in[a,b]} |A_1^*g_1(x)+\ldots+A_n^*g_n(x)-f(x)|,$$

where

$$a\le x_1^*<x_2^*<\ldots<x_{n+1}^*\le b.$$

Theorem 1.1. *Assume that $f(x)$, $g_1(x)$, $\dots$, $g_n(x)$ are continuously differentiable on $[a, b]$. Then each sequence $x_1^* < x_2^* \dots < x_{n+1}^*$ of the extremal points satisfies the system of equations*

$$\begin{cases} (x_1-a)D_1(x_1, \dots, x_{n+1}) = 0, & \\ D_i(x_1, \dots, x_{n+1}) = 0, & i = 2, 3, \dots, n \\ (x_{n+1}-b)D_{n+1}(x_1, \dots, x_{n+1}) = 0, & \end{cases}$$

where

$$D_k(x_1, \dots, x_{n+1}) = \begin{vmatrix} 0 & g_1'(x_k) & \dots g_n'(x_k) & f'(x_k) \\ 1 & g_1(x_1) & \dots g_n(x_1) & f(x_1) \\ -1 & g_1(x_2) & \dots g_n(x_2) & f(x_2) \\ \vdots & \vdots & \vdots & \\ (-1)^n & g_1(x_{n+1}) & \dots g_n(x_{n+1}) & f(x_{n+1}) \end{vmatrix}.$$

Proof. We can see in [3] that

(1) if $x_1^* \neq a$ and $x_{n+1}^* \neq b$, then

$$\begin{cases} (-1)^{i+1}C_1 + C_2 g_1(x_i^*) + \dots + C_{n+1} g_n(x_i^*) + f(x_i^*) = 0, & i = 1, 2, \dots, n+1 \\ C_2 g_1'(x_i^*) + \dots + C_{n+1} g_n'(x_i^*) + f'(x_i^*) = 0, & i = 1, 2, \dots, n+1 \end{cases}$$

(2) if $x_1^* = a$ and $x_{n+1}^* \neq b$, then

$$\begin{cases} (-1)^{i+1}C_1 + C_2 g_1(x_i^*) + \dots + C_{n+1} g_n(x_i^*) + f(x_i^*) = 0, & i = 1, 2, \dots, n+1 \\ C_2 g_1'(x_i^*) + \dots + C_{n+1} g_n'(x_i^*) + f'(x_i^*) = 0, & i = 2, 3, \dots, n+1 \end{cases}$$

(3) if $x_1^* \neq a$ and $x_{n+1}^* = b$, then

$$\begin{cases} (-1)^{i+1}C_1 + C_2 g_1(x_i^*) + \dots + C_{n+1} g_n(x_i^*) + f(x_i^*) = 0, & i = 1, 2, \dots, n+1 \\ C_2 g_1'(x_i^*) + \dots + C_{n+1} g_n'(x_i^*) + f'(x_i^*) = 0, & i = 1, 2, \dots, n \end{cases}$$

(4) if $x_1^* = a$ and $x_{n+1}^* = b$, then

$$\begin{cases} (-1)^{i+1}C_1 + C_2 g_1(x_i^*) + \dots + C_{n+1} g_n(x_i^*) + f(x_i^*) = 0, & i = 1, 2, \dots, n+1 \\ C_2 g_1'(x_1^*) + \dots + C_{n+1} g_n'(x_i^*) + f'(x_i^*) = 0, & i = 2, 3, \dots, n \end{cases}$$

If we determine $c_1, c_2, \dots, c_{n+1}$ from the system of the first $n+1$ equations of (1), (2), (3), (4) and we use these values in the remainder equations of (1), (2), (3), (4), then we get

$$(1) \qquad g_1'(x_i^*)\, d_1(x_1^*, \dots, x_{n+1}^*) + \dots + g_n'(x_i^*) d_n(x_1^*, \dots, x_{n+1}^*) = \\ = f(x_i^*) d(x_1^*, \dots, x_{n+1}^*), \qquad i = 1, 2, \dots, n+1$$

(2) $$g_1'(x_i^*)d_1(a, x_2^*, \ldots, x_{n+1}^*) + \ldots + g_n'(x_i^*)d_n(a, x_2^*, \ldots, x_{n+1}^*) = \\ = f'(x_i^*)d\,(a, x_2^*, \ldots, x_{n+1}^*), \qquad i = 2, 3, \ldots, n+1$$

(3) $$g_1'(x_i^*)d_1\,(x_1^*, \ldots, x_n^*, b) + \ldots + g_n'(x_i^*)\,d_n(x_1^*, \ldots, x_n^*, b) = \\ = f'(x_i^*)\,d(x_1^*, \ldots, x_n^*, b), \qquad i = 1, 2, \ldots, n$$

(4) $$g_1(x_i^*)\,d_1\,(a, x_2^*, \ldots, x_n^*, b) + \ldots + g_n'(x_i^*)\,d_n(a, x_2^*, \ldots, x_n^*, b) = \\ = f'(x_i^*)\,d(a, x_2^*, \ldots, x_n^*, b), \qquad i = 2, 3, \ldots, n,$$

where

$$d(y_1, \ldots, y_{n+1}) = \begin{vmatrix} 1 & g_1(y_1) & \ldots & g_n(y_1) \\ -1 & g_1(y_2) & \ldots & g_n(y_2) \\ \vdots & \vdots & & \vdots \\ (-1)^n & g_1(y_{n+1}) & \ldots & g_n(y_{n+1}) \end{vmatrix},$$

$$d_1(y_1, \ldots, y_{n+1}) = \begin{vmatrix} 1 & f(y_1) & g_2(y_1) & \ldots & g_n(y_1) \\ -1 & f(y_2) & g_2(y_2) & \ldots & g_n(y_2) \\ \vdots & & \vdots & \vdots & \vdots \\ (-1)^n & f(y_{n+1}) & g_2(y_{n+1}) & \ldots & g_n(y_{n+1}) \end{vmatrix}, \ldots,$$

$$d_n(y_1, \ldots, y_{n+1}) = \begin{vmatrix} 1 & g_1(y_1) & \ldots & f(y_1) \\ -1 & g_1(y_2) & \ldots & f(y_2) \\ \vdots & & & \\ (-1)^n & g_1(y_{n+1}) & \ldots & f(y_{n+1}) \end{vmatrix}.$$

If we sum up the determinants in the former equations, then we get

(1) $$D_i(x_1^*, \ldots, x_{n+1}^*) = 0, \qquad i = 1, 2, \ldots, n+1$$

(2) $$D_i(a, x_2^*, \ldots, x_{n+1}^*) = 0, \qquad i = 2, 3, \ldots, n+1$$

(3) $$D_i(x_1^*, \ldots, x_n^*, b) = 0, \qquad i = 1, 2, \ldots, n$$

(4) $$D_i(a, x_2^*, \ldots, x_n^*, b) = 0, \qquad i = 2, 3, \ldots, n.$$

The system of equations of our theorem "summarizes" the four cases. □

Theorem 1.2. *Let* $g_1(x) \equiv 1$, $g_2(x) \equiv x$, $\ldots$, $g_n(x) \equiv x^{n-1}$, *where* $n \geq 2$. *Assume that* $f(x)$ *is thrice-continuously differentiable and* $f^{(n)}(x)$ *exists on* $[a, b]$. *If* $f^{(n)}(x) \neq 0$, $\forall\, x \in (a, b)$, *then* $x_1^* = a$, $x_{n+1}^* = b$ *and the iteration*

$$x_2^* = x_2^* - \frac{\begin{vmatrix} 0 & \dots & g_i'(x_2^*) & \dots & f'(x_2^*) \\ 1 & \dots & g_i(x_1^*) & \dots & f(x_1^*) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(x_{n+1}^*) & \dots & f(x_{n+1}^*) \end{vmatrix}}{\begin{vmatrix} 0 & \dots & g_i''(x_2^*) & \dots & f''(x_2^*) \\ 1 & \dots & g_i(x_1^*) & \dots & f(x_1^*) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(x_{n+1}^*) & \dots & f(x_{n+1}^*) \end{vmatrix}}, \dots$$

$$\dots, x_n^* = x_n^* - \frac{\begin{vmatrix} 0 & \dots & g_i'(x_n^*) & \dots & f'(x_n^*) \\ 1 & \dots & g_i(x_1^*) & \dots & f(x_1^*) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(x_{n+1}^*) & \dots & f(x_{n+1}^*) \end{vmatrix}}{\begin{vmatrix} 0 & \dots & g_i''(x_n^*) & \dots & f''(x_n^*) \\ 1 & \dots & g_i(x_1^*) & \dots & f(x_1^*) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(x_{n+1}^*) & \dots & f(x_{n+1}^*) \end{vmatrix}}$$

gives the unique solution $x_2^*, \dots, x_n^*$ of our problem, if we start from "a sufficiently short distance". The rate of convergence of the iteration ≥ 2.

Proof. (1) If the error function $e(x) = A_1^* g_1(x) + \dots + A_n^* g_n(x) - f(x)$ of the best approximation has at least n extremal points on (a, b), then $e'(x) = 0$ has at least n roots on $(a, b) \Rightarrow e''(x) = 0$ has at least $n-1$ roots on $(a, b) \Rightarrow \dots \Rightarrow e^{(n)}(x) \equiv f^{(n)}(x) = 0$ has at least one root on (a, b). Hence $x_1^* = a$, $x_{n+1}^* = b$ and the set $\{x_2^*. \dots, x_n^*\}$ is unique.

(2) By Theorem 1.1 the extremal points $x_2^*, \dots, x_n^*$ satisfy the system of equations

$$\begin{vmatrix} 0 & \dots & g_i'(x_k) & \dots & f'(x_k) \\ 1 & \dots & g_i(a) & \dots & f(a) \\ -1 & \dots & g_i(x_2) & \dots & f(x_2) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(b) & \dots & f(b) \end{vmatrix} = 0, \ k = 2, 3, \dots, n.$$

The elements of the Jacobi matrix J of this system of equations:

$$J_{11} = \begin{vmatrix} 0 & \dots & g_i''(x_2) & \dots & f''(x_2) \\ 1 & \dots & g_i(a) & \dots & f(a) \\ -1 & \dots & g_i(x_2) & \dots & f(x_2) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(b) & \dots & f(b) \end{vmatrix},$$

$$J_{12} = \begin{vmatrix} 0 & \dots & g_i'(x_2) & \dots & f'(x_2) \\ \vdots & & & & \vdots \\ -1 & \dots & g_i(x_2) & \dots & f(x_2) \\ 0 & \dots & g_i'(x_3) & \dots & f'(x_3) \\ -1 & \dots & g_i(x_4) & \dots & f(x_4) \\ \vdots & & & & \vdots \end{vmatrix}, \dots .$$

Now we shall prove that $J_{11}(x_2^*, \dots, x_n^*) \neq 0$ and $J_{12}(x_2^*, \dots, x_n^*) = 0$ (generally $J_{m,m}(x_2^*, \dots, x_n^*) \neq 0$ and $J_{m,j}(x_2^*, \dots, x_n^*) = 0$ if $m \neq j$, where $1 \leq m, j \leq n-1$).

Since $x_2^*, \dots, x_n^*$ are roots of the function

$$s(x) = \begin{vmatrix} 0 & \dots & g_i'(x) & \dots & f'(x) \\ 1 & \dots & g_i(a) & \dots & f(a) \\ -1 & \dots & g_i(x_2^*) & \dots & f(x_2^*) \\ \vdots & & & & \\ (-1)^n & \dots & g_i(b) & \dots & f(b) \end{vmatrix} = A_1 g_1'(x) + A_2 g_2'(x) + \dots + A_n g_n'(x) + A f'(x),$$

therefore if $s'(x_2^*) = J_{11}(x_2^*, \dots, x_n^*) = 0$, then the function $s'(x) = A_1 g_1''(x) + A_2 g_2''(x) + \dots + A_n g''(x) + Af''(x)$ has at least $n-1$ roots on $(a, b) \Rightarrow s^{(n-1)}(x) = Af^{(n)}(x)$ has at least one root on (a, b). As $A \neq 0$ $J_{11}(x_2^*, \dots, x_n^*) \neq 0$ (generally $J_{m,m}(x_2^*, \dots, x_n^*) \neq 0$).

Our last system of equations springs from the system of equations

$$\begin{cases} (E_1) & C_1 + C_2\, g_1(a) + \dots + C_{n+1}\, g_n(a) + f(a) = 0 \\ (E_2) & -C_1 + C_2\, g_1(x_2^*) + \dots + C_{n+1}\, g_n(x_2^*) + f(x_2^*) = 0 \\ \vdots & \vdots \\ (E_{n+1})\ (-1)^{n+2} & C_1 + C_2\, g_1(b) + \dots + C_{n+1}\, g_n(b) + f(b) = 0 \\ (E_{n+2}) & C_2\, g_1'(x_2^*) + \dots + C_{n+1}\, g_n'(x_2^*) + f'(x_2^*) = 0 \\ \vdots & \vdots \\ (E_{2n}) & C_2\, g_1'(x_n^*) + \dots + C_{n+1}\, g_n'(x_n^*) + f'(x_n^*) = 0 \end{cases}$$

(see the proof of Theorem 1.1 and [3]). Determining $C_1, C_2, \ldots, C_{n+1}$ from the system of equations (E_1), (E_2), (E_{n+3}), (E_4), $\ldots$, (E_{n+1}) and using these values in (E_{n+2}) we get (after summarizing of the determinants) $J_{12}(x_2^*, \ldots, x_n^*) = 0$ (similarly $J_{m,j}(x_2^*, \ldots, x_n^*) = 0$, $m \neq j$).

Hence

$$J(x_2^*, \ldots, x_n^*) = \begin{bmatrix} J_{11}(x_2^*, \ldots, x_n^*) & & \\ & \ddots & 0 \\ 0 & & J_{n-1, n-1}(x_2^*, \ldots, x_n^*) \end{bmatrix}$$

is a nonsingular matrix. Let us see the well-known formula

$$\begin{bmatrix} J_{11}(x_2, \ldots, x_n) \ldots J_{1, n-1}(x_2, \ldots, x_n) \\ \vdots \qquad\qquad \vdots \\ J_{n-1,1}(x_2, \ldots, x_n) \ldots J_{n-1, n-1}(x_2, \ldots, x_n) \end{bmatrix} \begin{bmatrix} \tilde{x}_2 - x_2 \\ \vdots \\ \tilde{x}_n - x_n \end{bmatrix} =$$

$$= \begin{bmatrix} |0 \ldots g_i'(x_2) \ldots f'(x_2)| \\ \vdots \\ |0 \ldots g_i'(x_n) \ldots f'(x_n)| \end{bmatrix}$$

of the Newton's method (where $x = \{x_2, \ldots, x_n\}$ and $\tilde{x} = \{\tilde{x}_2, \ldots, \tilde{x}_n\}$ are "old and new approach" of $x^* = \{x_2^*, \ldots, x_n^*\}$).

If we use in place of the matrix J the matrix

$$J^* = \begin{bmatrix} J_{11} & & \\ & \ddots & 0 \\ 0 & & J_{n-1, n-1} \end{bmatrix},$$

then we get the iteration of our theorem. Since $J^*(x_2^*, \ldots, x_n^*) = J(x_2^*, \ldots$ $\ldots, x_n^*)$ and the functions of our system of equations are at least twice-continuously differentiable in a sufficiently short distance to $\{x_2^*, \ldots, x_n^*\}$, therefore our iteration satisfies Theorem 1.2. □

2. A program to the iteration of Theorem 1.2

The input functions and parameters of our BASIC program:

100: $f(x) \sim F(x)$
110: $f'(x) \sim D1(x)$
120: $f''(x) \sim D2(x)$
130: $n \sim N$, $a \sim AA$, $b \sim BB$, $\varepsilon \sim EP$ (the iteration stops if $\|\tilde{x} - x\|_\infty < \varepsilon$)
140: starting values to $x_2^*, \ldots, x_n^* \sim X(2), \ldots, X(N)$

The output parameters:

430: $x_2^*, \ldots, x_n^* \sim X(2), \ldots, X(N)$.

The complete program (to the first example) is as follows.

```
100 DEFFNF(X)=EXP(X)
110 DEFFND1(X)=EXP(X)
120 DEFFND2(X)=EXP(X)
130 INPUTN,AA,BB,EP
140 FORI=2TON:INPUTX(I):NEXTI
150 A(2,1)=1:A(2,2)=1:M=N+2
160 FORI=3TON+1:A(2,I)=AA ↑ (I-2):NEXTI
170 A(2,N+2)=FNF(AA):A(N+2,N+2)=FNF(BB)
180 A(N+2,1)=(-1) ↑ N:A(N+2,2)=1
190 FORI=3TON+1:A(N+2,I)=BB ↑ (I-2):NEXTI
200 FORI=3TON+1
210 A(I,1)=(-1) ↑ I:A(I,N+2)=FNF(X(I-1))
220 FORJ=2TON+1:A(I,J)=X(I-1) ↑ (J-2):NEXTJ:NEXTI
230 FORJ=MTO3STEP-1:MA=0:FORI=JTO2STEP-1
240 IF(ABS(A(I,J))>MA)THENMA=ABS(A(I,J)):K=I
250 NEXTI:IFK=JTHEN280
260 FORI=1TOM
270 B(I)=-A(K,I):A(K,I)=A(J,I):A(J,I)=B(I):NEXTI
280 FORI=J-1TO2STEP-1:FORK=1TOJ
290 A(I,K)=A(I,K)-A(J,K)*A(I,J)/A(J,J)
300 NEXTK:NEXTI:NEXTJ:FORL=2TON
310 A(1,1)=0:A(1,2)=0:A(1,3)=1
320 FORI=4TON+1
330 A(1,I)=X(L) ↑ (I-3)*(I-2):NEXTI
340 A(1,N+2)=FND1(X(L)):GOSUB1000
350 A(1,1)=0:A(1,2)=0:A(1,3)=0:C1(L)=DE
360 FORI=4TON+1
370 A(1,I)=X(L) ↑ (I-4)*(I-2)*(I-3):NEXTI
380 A(1,N+2)=FND2(X(L))
390 GOSUB1000:C2(L)=DE:NEXTL:FORL=2TON
400 Z(L)=X(L):X(L)=X(L)-C1(L)/C2(L):MA=0
410 IF(ABS(X(L)-Z(L))>MA)THENMA=ABS(X(L)-Z(L))
420 NEXTL:IF(MA>EP)THEN150
430 PRINT:PRINT:FORI=2TON:PRINTX(I):NEXTI:END
1000 FORJ=MTO2STEP-1:FORK=1TOJ
1010 A(1,K)=A(1,K)-A(J,K)*A(1,J)/A(J,J)
1020 NEXTK:NEXTJ:DE=1
1030 FORJ=1TOM:DE=DE*A(J,J):NEXTJ:RETURN
```

We tested the program on a COMMODORE-64 computer and determined the starting values of $x_2^*, \ldots, x_n^*$ from the formula

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\pi \frac{n-i+1}{n}\right), \quad i = 2, 3, \ldots, n$$

(see [1]). We used the program to the following 12 examples:

$$f(x) = e^x, \qquad [a, b] = [0, 1], \qquad n = 2, 3, 4, 5,$$

$$f(x) = \ln x, \qquad [a, b] = [1, e], \qquad n = 2, 3, 4, 5,$$

$$f(x) = \sin x, \qquad [a, b] = [0, \pi/4], \qquad n = 2, 3, 4, 5,$$

The iteration is convergent in every case and finally we get the errors of the best Chebyshev approximations:

$\sim$ 0.1,	0.009,	0.0006,	0.00003
$\sim$ 0.06,	0.01,	0.002,	0.0004
$\sim$ 0.02,	0.002,	0.00005,	0.000005.

(We used two "helping programs". The first program determines $A_1^*, \ldots, A_n^*$ by solving a linear system of equations and the second program gives the figure of the error curve).

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