ON L_1-MEAN OSCILLATING RANDOM VARIABLES

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1. The $\mathcal{K}_p$-spaces are treated e.g. in the book by A. M. Garsia [1]. Let $X \in L^1(\Omega, \mathcal{A}, P)$ be a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$ and consider the regular martingale

$$X_n = E(X \mid \mathcal{F}_n), \ n \geq 0,$$

where $\{\mathcal{F}_n\}$, $n \geq 0$, is an increasing sequence of σ-fields of events such that

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n \right) = \mathcal{A}.$$

We suppose that $X_0 = 0$ a.s. We denote by $d_0 = 0, d_1, d_2, \ldots$ the difference sequence corresponding to the martingale $(X_n, \mathcal{F}_n)$.

For $1 \leq p \leq +\infty$ set

$$\Gamma_X^{(p)} = \{\gamma\colon \gamma \in L_p E, \ (|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a. s.}, \ \forall\, n \geq 1\}.$$

We say that $X \in \mathcal{K}_p$ if the set $\Gamma_X^{(p)}$ is not empty and in this case we let

$$\|X\|_{\mathcal{K}_p} = \inf_{\gamma \in \Gamma_X^{(p)}} \|\gamma\|_p.$$

It easily can be seen that $\|\cdot\|_{\mathcal{K}_p}$ is a semi-norm on $\mathcal{K}_p$. The space $\mathcal{K}_\infty$ is the well-known BMO_1-space.

In [2] we generalized this notion in the following way. Consider a pair (Φ, Ψ) of conjugate Young functions and put

$$\Gamma_X^{(\Phi)} = \{\gamma\colon \gamma \in L^\Phi, \ E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a. s.}, \ \forall\, n \geq 1\}.$$

We say that $X \in \mathcal{K}_\Phi$ if the set $\Gamma_X^{(\Phi)}$ is not empty. In this case we let

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Gamma_X^{(\Phi)}} \|\gamma\|_\Phi,$$

where $\|\cdot\|_\Phi$ denotes the Luxemburg norm in the Orlicz space L^Φ. For the definition of the Young functions, Orlicz spaces and Luxemburg norms we refer to [3] and [4]. We easily prove that $\|\cdot\|_{\mathcal{K}_\Phi}$ is a semi-norm on $\mathcal{K}_\Phi$.

We say that the random variable X belongs to the Hardy space $\mathcal{H}_\Phi$ if

$$S=S(X)=\left(\sum_{i=1}^{\infty} d_i^2\right)^{1/2}\in L^\Phi,$$

or in other words $\|S\|_\Phi<+\infty$. In this case we write $\|X\|_{\mathcal{H}_\Phi}=\|S\|_\Phi$.

Since the Young functions Φ cannot be linear, the space L_1 is not contained amongst the Orlicz spaces. Therefore we define the Hardy space $\mathcal{H}_1$ as the set of all the random variables X for which $\|S\|_1<+\infty$. In this case we let $\|X\|_{\mathcal{H}_1}=\|S\|_1$.

We recall the definition of the power of a Young function Φ. Let $\varphi(x)$ be the right hand side derivative of Φ. Then the quantity

$$p=\sup_{x>0}\frac{x\varphi(x)}{\varphi(x)}$$

is called the power of Φ. The finiteness of p is equivalent to say that Φ satisfies the so called Δ_2-condition. We define similarly the power q of the conjugate Young function $\Psi(x)$.

The inequality of Burkholder-Davis-Gundy says that if p is finite then $X\in\mathcal{H}_\Phi$ if and only if $X^*=\sup_{n\geq 0}|X_n|\in L^\Phi$ (cf. [5]). Also, Davis' inequality states that $X\in\mathcal{H}_1$ if and only if $X^*\in L^1$.

In paper [2] we proved if both Φ and Ψ have finite power then $X\in\mathcal{K}_\Phi$ is equivalent to the fact that $X\in\mathcal{H}_\Phi$, or in other words $X^*\in L^\Phi$. For the pair $\left(\frac{X^p}{p},\frac{X^q}{q}\right)$ of conjugate Young functions, where $p>1$ and $p^{-1}+q^{-1}=1$ this fact has essentially been established by Garsia in [1] (Theorem III. 5.2.).

The space $\mathcal{K}_1$ is less studied. We only know that $\mathcal{H}_1\subset\mathcal{K}_1$. In fact, if $X^*\in L^1$, then

$$E(|X-X_{n-1}|\,|\mathcal{F}_n)\leq E(2X^*|\mathcal{F}_n),\ n\geq 1.$$

Consequently, $\Gamma_X^{(1)}$ is not empty, since $X^*\in L^1$ and so $2X^*\in\Gamma_X^{(1)}$. The reverse implication, i.e. $\mathcal{K}_1\subset\mathcal{H}_1$, is false. Here is a counterexample. Consider a non-negative random variable X belonging to L_1. Let $X_n=E(X|\mathcal{F}_n)$, $n\geq 0$ be the corresponding martingale. Also let $X_n'=X_n-X_0$, $n\geq 0$. Then $(X_n',\mathcal{F}_n)$ is also a martingale. Suppose we have chosen such an X that the limit $X-X_0$ of X_n-X_0 does not belong to $\mathcal{H}_1$ but at the same time $|X_{n+1}'-X_n'|\leq 1$ a.s. We show that $X-X_0$ belongs to $\mathcal{K}_1$. In fact,

$$E(|X-X_0-X_{n-1}'|\,|\mathcal{F}_n)=E(|X-X_{n-1}|\,|\mathcal{F}_n)\leq E(|X-X_n|\,|\mathcal{F}_n)+$$
$$+|X_n-X_{n-1}|\leq E(X|\mathcal{F}_n)+E(X|\mathcal{F}_n)+|X_n'-X_{n-1}'|\leq E(2X+1/\mathcal{F}_n),$$

which shows that $X-X_0\notin\mathcal{H}_1$ and that $X-X_0\in\mathcal{K}_1$ (cf. e.g. [1], p. 122.).

In what follows we shall use a maximal inequality which is proved in [2]. We state it in the form of the following

Theorem 1. *Let $(X_n, \mathcal{F}_n)$ be a martingale and let $\gamma \in L^1$ be a random variable such that for every $n \geq 1$ we have*

$$E(|X - X_{n-1}| \, |\mathcal{F}_n) \leq E(\gamma | \mathcal{F}_n) \text{ a.s.}$$

Then for arbitrary $\beta > \alpha > 0$ we have

$$(\beta - \alpha) E(\chi(X_n^* \geq \beta)) \leq E(\gamma \chi(X_n^* \geq \alpha)).$$

Here $\chi(A)$ denotes the indicator function of the event A.

2. About the behaviour of the random variables belonging to $\mathcal{X}_1$ we can prove the following

Theorem 2. *If $X \in \mathcal{X}_1$ then X^* is a.s. finite. Moreover, for arbitrary $\lambda > 0$ we have the inequality*

$$\lambda P(X^* \geq \lambda) \leq 2 \|X\|_{\mathcal{X}_1}.$$

Proof. We use the inequality of Theorem 1. According to this if $\beta > \alpha > 0$ and if $X \in \mathcal{X}_1$ then with arbitrary $\gamma \in \Gamma_X^{(1)}$ we have

$$(\beta - \alpha) E(\chi(X^* \geq \beta)) \leq E(\gamma \chi(X^* \geq \alpha)).$$

Choose $\beta = 2\alpha$. Then

$$\alpha P(X^* \geq 2\alpha) \leq E(\gamma).$$

Since $\gamma \in \Gamma^{(1)}$ is arbitrary from this we get

$$\alpha P(X^* \geq 2\alpha) \leq \|X\|_{\mathcal{X}_1},$$

or, in other words

$$2\alpha P(X^* \geq 2\alpha) \leq 2 \|X\|_{\mathcal{X}_1}.$$

Taking $\lambda = 2\alpha$ we obtain our inequality.

Further, since

$$P(X^* = +\infty) = \lim_{\lambda \to +\infty} P(X^* \geq \lambda),$$

the inequality just proved shows that

$$P(X^* = +\infty) \leq 2 \|X\|_{\mathcal{X}_1} \lim_{\lambda \to +\infty} \frac{1}{\lambda} = 0.$$

This means that $P(X^* < +\infty) = 1$. □

3. When at least one of Φ and Ψ have no finite power then we cannot prove the equivalence of the norms $\|\cdot\|_{\mathcal{X}_\Phi}$ and $\|\cdot\|_{\mathcal{H}_\Phi}$. Assuming only the finiteness of the power p of Φ we are able to prove the validity of the following inequality: if $X \in \mathcal{X}_\Phi$, $P(X = 0) < 1$, then with arbitrary constants $c > 1$ and $\varrho > 1$ we have

$$(\varrho - 1) E\left(\Psi\left(\varphi\left(\frac{X_n^*}{\varrho A \frac{c}{c-1} \|X\|_{\mathcal{X}_\Phi}} \right)\right)\right) \leq 1.$$

Here $A = A(c)$ is the number for which

$$\varphi(ct) \leqq A\varphi(t)$$

is satisfied for every $t > 0$.

In these considerations the Young function Φ is „far" from the linear function. Namely, $\Phi(x)/x$ tends increasingly to $+\infty$ as $x \uparrow +\infty$. So, it is of interest to consider separately the case of the linear function, too. In analogy with the classical result of Doob stating that $X \in \mathcal{H}_1$ whenever $X \in L \log L$ we can deduce the following

Theorem 3. *Suppose that $X \in \mathcal{K}_\Phi$, where $\Phi(x) = x \log^+ x$. Then $X \in \mathcal{H}_1$ and we have*

$$E(X_n^*) \leq \frac{12e}{e-2}(e + \|X\|_{\mathcal{K}_{x\log^+ x}}) \log (e + \|X\|_{\mathcal{K}_{x\log^+ x}}).$$

Proof. For the proof we use the inequality of Theorem 1. Choosing $\beta = 2\alpha$ we get

$$\alpha E(\chi(x_n^* \geq 2\alpha)) \leq E(\gamma\chi(x_n^* \geq \alpha)),$$

where $\gamma \in \Gamma_X^{(\Phi)}$ is arbitrary. Multiply this inequality by $1/\alpha$ and integrate with respect to α on the interval $[1, +\infty)$. Then

$$E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E(\gamma \log^+ X_n^*),$$

or, in other words,

$$\frac{1}{\max(e, \|\gamma\|_\Phi)} E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E\left(\frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ X_n^*\right).$$

Using the elementary inequality

$$a \log^+ b \leq a \log^+ a + \frac{b}{e}$$

which is valid for arbitrary $a \geq 0$ and $b \geq 0$ we obtain on the right-hand side

$$\frac{1}{\max(e, \|\gamma\|_\Phi)} E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E\left(\frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ \frac{\gamma}{\max(e, \|\gamma\|_\Phi)} + \right.$$

$$\left. + \frac{X_n^*}{e \max(e, \|\gamma\|_\Phi)} + \frac{\gamma}{\max(e, \|\gamma\|_\Phi)} \log^+ \max(e, \|\gamma\|_\Phi)\right).$$

Note that

$$\frac{X_n^*}{2} \leq \left(\frac{X_n^*}{2} - 1\right)^+ + 1.$$

From this and from the above inequlity

$$\frac{1}{2\max(e, \|\gamma\|_\Phi)} E(X_n^*) \leq 2 + \frac{1}{e\max(e, \|\gamma\|_\Phi)} E(X_n^*) + $$
$$+ \frac{E(\gamma)\log\max(e, \|\gamma\|_\Phi)}{\max(e, \|\gamma\|_\Phi)},$$

since

$$\log^+ \max(e, \|\gamma\|_\Phi) = \log\max(e, \|\gamma\|_\Phi).$$

This implies that

$$\frac{e-2}{2e} E(X_n^*) \leq 2\max(e, \|\gamma\|_\Phi) + E(\gamma)\log\max(e, \|\gamma\|_\Phi).$$

Now we show that

$$E(\gamma) \leq 4\|\gamma\|_\Phi.$$

In fact, in case of any Young function Φ we have for every $x>0$ the inequality

$$\Phi(x) \geq (x-x_0)\varphi(x_0),$$

where $x_0>0$ satisfies $\varphi(x_0)>0$. Here, as usual, φ denotes the right-hand side derivative of Φ. Consequently, for arbitrary $Y \in L^\Phi$ such that $P(Y=0)<0$ we see that

$$1 \geq E\left(\Phi\left(\frac{|Y|}{\|Y\|_\Phi}\right)\right) \geq \varphi(x_0) E\left(\left(\frac{|Y|}{\|Y\|_\Phi} - x_0\right)^+\right).$$

Using the inequality $x \leq (x-x_0)^+ + x_0$ from the preceding inequality we get

$$E\left(\frac{|Y|}{\|Y\|_\Phi}\right) \leq \frac{1}{\varphi(x_0)} + x_0,$$

or, in other words

$$E(|Y|) \leq \left(\frac{1}{\varphi(x_0)} + x_0\right)\|Y\|_\Phi.$$

Now, turning to our case, we have $\varphi(x) = 1+\log x$, if $x \geq 1$ and $\varphi(x) = 0$ if $x<1$. Consequently, choosing $x_0 = e$ we get $\varphi(x_0) = 1+\log e = 2$. So,

$$E(\gamma) \leq \left(\frac{1}{2} + e\right)\|\gamma\|_\Phi \leq 4\max(e, \|\gamma\|_\Phi).$$

Comparing this with the inequality above we get

$$\frac{e-2}{2e} E(X_n^*) \leq 6\max(e, \|\gamma\|_\Phi)\log\max(e, \|\gamma\|_\Phi),$$

which implies the inequality

$$E(X_n^*) \leq \frac{12e}{e-2}\Big((e+\|X\|_{\chi_\Phi}) \log (e+\|X\|_{\chi_\Phi}\Big).$$

This proves the assertion. □

Remark. $\Phi(x) = x \log^+ x$ will be a Young function only in the case when we define its right hand side derivative $\varphi(x)$ to be right continuous at $x = 1$. This means that $\varphi(+1)$ must be equal to 1.

As usual, we say that $X \in L \log L$ if $E(|X| \log^+ |X|) < +\infty$. Consider the set

$$\Gamma_X'^{(x \log^+ x)} = \{\gamma : \gamma \in L \log L, E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leqq E(\gamma) | \mathcal{F}_n) \text{ a. s. } \forall\, n \geqq 1\}.$$

Then $\Gamma_X'^{(x \log^+ x)}$ is a subset of $\Gamma_X^{(x \log^+ x)}$ and we have

$$\|\gamma\|_{x \log^+ x} \leq \max\big(1, E(\gamma \log^+ \gamma)\big).$$

In fact, if $\gamma \in \Gamma_X'^{(X \log^+ X)}$, then $E(\gamma \log^+ \gamma) < +\infty$. Consequently, if $E(\gamma \log^+ \gamma) > 1$ then by the convexity of $\Phi(x) = x \log^+ x$,

$$E\left(\frac{\gamma}{E(\gamma \log^+ \gamma)} \log^+ \frac{\gamma}{E(\gamma \log^+ \gamma)}\right) \leq \frac{1}{E(\gamma \log^+ \gamma)} E(\gamma \log^+ \gamma) = 1.$$

If, conversely, $E(\gamma \log^+ \gamma) \leq 1$, then trivially $\|\gamma\|_{x \log^+ x} \leq 1$. Therefore,

$$\|\gamma\|_{x \log^+ x} \leqq \max\big(1, \mathrm{E}(\gamma \log^+ \gamma)\big). \quad □$$

It seems to be interesting to deduce an inequality, like the preceding one for $E(X_n^*)$ in case of the class $\Gamma_X'^{(x \log^+ x)}$. It can be shown that in this case the inequality to be proved is simpler than that of the preceding assertion.

Theorem 4. *Let X be a random variable and suppose that the set $\Gamma_X'^{(x \log^+ x)}$ defined by the formula*

$$\Gamma_X'^{(x \log^+ x)} = \{\gamma : \gamma \in L \log L, E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leq E(\gamma | \mathcal{F}_n) \text{ a. s. } \forall\, n \geq 1\}.$$

is not empty. Then $X \in \mathcal{H}_1$ and we have

$$E(X_n^*) \leq \frac{e-2}{2e}\Big(1 + \inf_{\gamma \in \Gamma_X'^{(x \log^+ x)}} E(\gamma \log^+ \gamma)\Big).$$

Proof. Again, we shall use the inequality

$$(\beta - \alpha) E\big(\chi(X_n^* \geq \beta)\big) \leq E\big(\gamma \chi(X_n^* \geq \alpha)\big)$$

and we choose $\beta = 2\alpha$. Here $\gamma \in \Gamma_X'^{(x \log^+ x)}$ is arbitrary. Then

$$\alpha E\left(\chi\left(\frac{X_n^*}{2} \geq \alpha\right)\right) \leq E\big(\gamma \chi(X_n^* \geq \alpha)\big).$$

Integrate this with respect to the measure $d\alpha/\alpha$ on the interval $[1, +\infty)$. We then get

$$E\left(\left(\frac{X_n^*}{2} - 1\right)^+\right) \leq E(\gamma \log^+ X_n^*) \leq E\left(\gamma \log^+ \gamma + \frac{X_n^*}{e}\right).$$

From this

$$\frac{e-2}{2e}E(X_n^*) \leq 1 + E(\gamma \log^+ \gamma)$$

and finally

$$E(X_n^*) \leq \frac{2e}{e-2}\left[1 + \inf_{\gamma \in \Gamma_X'(x \log^+ x)} E(\gamma \log^+ \gamma)\right].$$

This proves the assertion. □

REFERENCES

[1] *Garsia A. M.:* Martingale Inequalities. Seminar Notes in Recent Progress. Benjamin, Reading (Massachusetts), 1973.

[2] *Bassily N. L.* and *Mogyoródi J.:* On the $\mathcal{K}_\Phi$-spaces with general Young function. *Annales Univ. Sci. Budapest., Sectio Mathematica.* **27** (1984), 205–214.

[3] *Neveu J.:* Discrete Parameter Martingales. North Holland, Amsterdam, 1975.

[4] *Krasnoselskii M. A.* and *Rutickii Ya. B.:* Convex Functions and Orlicz Spaces. Noordhoff, Groningen, 1961.

[5] *Burkholder D. L.:* Distribution function inequalities for martingales. *The Annals of Probability* **1** (1973), 19–42.