

(0,2,4) LACUNARY INTERPOLATION WITH G-SPLINES

THARWAT FAWZY

Math. Dept., Fac. of Science, Suez-Canal Univ.
Ismailia, Egypt

(Received August 5, 1985)

1. Introduction. Recently, R. S. Mishra and K. K. Mathur [1] introduced a global method for solving the (0, 2, 4) interpolation problem with splines. They constructed spline polynomial of degree 6 and deficiency 2 for functions $f \in C^5$. They introduced the following convergence theorem.

Theorem 1.1. (Mishra and Mathur). *Let $f \in C^5[0, 1]$ and $S_n(x)$ be the unique spline satisfying the conditions of Theorem 3 of [1]. Then $(n = 2m + 1)$*

$$\|S_n^{(q)}(x) - f^{(q)}(x)\|_{\infty} \leq 2245 m^{q-5} \omega_5\left(\frac{1}{m}\right) + 4m^{q-5} \|f^{(5)}\|_{\infty}$$

where $q = 0, 1, 2, 3, 4$.

In this paper we study the following interpolation problem.

Problem. Given $\Delta: \{x_i = ih\}_{i=0}^n$ and arbitrary real numbers $\{f_i, f_i', f_i'', f_i^{(iv)}\}_{i=0}^n$. Find S such that

$$(1.1) \quad S^{(q)}(x_i) = f^{(q)}(x_i) = f_i^{(q)}, \quad q = 0, 2, 4, \quad i = 0, 1, \dots, n. \quad \square$$

The purpose of this paper is to construct a local spline method for solving this problem using piece-wise polynomials of degree 6, such that for all functions $f \in C^6$, the order of approximation is the same as the best approximation using splines of degree 6.

2. Construction of the spline interpolant. We shall construct a solution S of the given problem in the form:

$$(2.1) \quad S_{\Delta}(x) = S_k(x) = \sum_{j=0}^6 \frac{S_k^{(j)}}{j!} (x - x_k)^j, \quad x_k \leq x \leq x_{k+1},$$

where $k = 0, 1, \dots, n-1$.

We shall define each of the $S_k^{(j)}$ explicitly in terms of the data. In particular we choose

$$(2.2) \quad S_k^{(q)} = f_k^{(q)}, \quad q = 0, 2, 4 \text{ and } k = 0, 1, \dots, n-1.$$

For $k = 1, 2, \dots, n-2$ we take

$$(2.3) \quad S_k^{(6)} = \frac{1}{h^2} \left\{ f_{k+1}^{(iv)} - 2f_k^{(iv)} + f_{k-1}^{(iv)} \right\},$$

$$(2.4) \quad S_k^{(5)} = \frac{1}{h} \left\{ f_{k+1}^{(iv)} - f_k^{(iv)} - \frac{h^2}{2} S_k^{(6)} \right\},$$

$$(2.5) \quad S_k^{(3)} = \frac{1}{h} \left\{ f_{k+1}'' - f_k'' - \frac{h^2}{2} f_k^{(iv)} - \frac{h^3}{3!} S_k^{(5)} - \frac{h^4}{4!} S_k^{(6)} \right\} \text{ and}$$

$$(2.6) \quad S_k^{(1)} = \frac{1}{h} \left\{ f_{k+1} - f_k - \frac{h^2}{2} f_k'' - \frac{h^3}{3!} S_k^{(3)} - \frac{h^4}{4!} f_k^{(iv)} - \frac{h^5}{5!} S_k^{(5)} - \frac{h^6}{6!} S_k^{(6)} \right\}.$$

For $k = 0$ we take

$$(2.7) \quad S_0^{(6)} = S_1^{(6)},$$

$$(2.8) \quad S_0^{(5)} = \frac{1}{h} \left\{ f_1^{(iv)} - f_0^{(iv)} - \frac{h^2}{2} S_0^{(6)} \right\},$$

$$(2.9) \quad S_0^{(3)} = \frac{1}{h} \left\{ f_1'' - f_0'' - \frac{h^2}{2} f_0^{(iv)} - \frac{h^3}{3!} S_0^{(5)} - \frac{h^4}{4!} S_0^{(6)} \right\} \text{ and}$$

$$(2.10) \quad S_0^{(1)} = \frac{1}{h} \left\{ f_1 - f_0 - \frac{h^2}{2} f_0'' - \frac{h^3}{3!} S_0^{(3)} - \frac{h^4}{4!} f_0^{(iv)} - \frac{h^5}{5!} S_0^{(5)} - \frac{h^6}{6!} S_0^{(6)} \right\}.$$

Finally for $k = n-1$ we take

$$(2.11) \quad S_{n-1}^{(6)} = S_{n-2}^{(6)},$$

$$(2.12) \quad S_{n-1}^{(5)} = \frac{1}{h} \left\{ f_n^{(iv)} - f_{n-1}^{(iv)} - \frac{h^2}{2} S_{n-1}^{(6)} \right\},$$

$$(2.13) \quad S_{n-1}^{(3)} = \frac{1}{h} \left\{ f_n'' - f_{n-1}'' - \frac{h^2}{2} f_{n-1}^{(iv)} - \frac{h^3}{3!} S_{n-1}^{(5)} - \frac{h^4}{4!} S_{n-1}^{(6)} \right\} \text{ and}$$

$$(2.14) \quad S_{n-1}^{(1)} = \frac{1}{h} \left\{ f_n - f_{n-1} - \frac{h^2}{2} f_{n-1}'' - \frac{h^3}{3!} S_{n-1}^{(3)} - \frac{h^4}{4!} f_{n-1}^{(iv)} - \frac{h^5}{5!} S_{n-1}^{(5)} - \frac{h^6}{6!} S_{n-1}^{(6)} \right\}.$$

The relation (2.3) is an assumed formula and it has been chosen to give a good approximation to $f^{(6)}(x)$, $x_k \leq x \leq x_{k+1}$.

The relations (2.4), (2.5) and (2.6) have been chosen to make S_Δ satisfying the following condition for $k = 1, 2, \dots, n-2$:

$$(2.15) \quad S_k^{(q)}(x_{k+1}) = S_{k+1}^{(q)}(x_{k+1}) = f_{k+1}^{(q)}, q = 0, 2, 4.$$

Similarly, the relations (2.8), (2.9) and (2.10) have been chosen to make S_Δ satisfying the condition (2.15) for $k = 0$, while the relation (2.7) has been chosen to make

$$(2.16) \quad D_L^6 S_0(x_1) = D_R^6 S_1(x).$$

where D_R is the right derivative.

Finally, the relation (2.11) has been chosen to make

$$(2.17) \quad D_L^6 S_{n-2}(x_{n-1}) = D_R^6 S_{n-1}(x_{n-1}),$$

while the relations (2.12), (2.13) and (2.14) have been chosen to make S_Δ satisfying the condition

$$(2.18) \quad S_\Delta^{(q)}(x_n) = f_n^{(q)}, q = 0, 2, 4.$$

Clearly, the function S defined in (2.1)–(2.14) solves the interpolation problem. Moreover, by construction it is clear that S is a piece-wise polynomial of degree 6.

Indeed S is the unique piece-wise polynomial of degree 6 in

$$(2.19) \quad C^{(0, 2, 4, 6)}[x_0, x_2] \cap C^{(0, 2, 4)}[x_0, x_n] \cap C^{(0, 2, 4, 6)}[x_{n-2}, x_n]$$

and satisfying the interpolation condition (1.1), where

$$(2.20) \quad C^{(0, 2, 4, 6)}[a, b] = \{f: f, D_R^2 f, D_R^4 f, D_R^6 f \in C[a, b]\} \text{ and}$$

$$(2.21) \quad C^{(0, 2, 4)}[a, b] = \{f: f, D_R^2 f, D_R^4 f \in C[a, b]\}.$$

S is a special kind of g -splines, we refer to it as lacunary g -spline.

3. Error bounds for the spline interpolant. Suppose $f \in C^6[x_0, x_n]$. Then using Taylor and dual Taylor expansions it is easy to establish the following lemma estimating how well the $S_k^{(j)}$ approximate $f^{(j)}(x_k)$ in term of the modulus of continuity $\omega(D^6 f; h)$ of $f^{(6)}(x)$.

Lemma 3.1. For $1 \leq k \leq n-2$ and $j = 1, 3, 5$ and 6, we have

$$(3.1) \quad |S_k^{(j)} - f^{(j)}(x_k)| \leq C_{jk} h^{6-j} \omega(D^6 f; h),$$

where the constants c_{jk} are the following: $c_{1k} = \frac{7}{240}$, $c_{3k} = \frac{5}{48}$, $c_{5k} = \frac{3}{4}$

and $c_{6k} = \frac{3}{2}$.

Theorem 3.1. Let $f \in C^6[x_0, x_n]$ and let S_Δ be the lacunary g -spline constructed in (2.1)–(2.14). Then for all $0 \leq j \leq 6$ and all $1 \leq k \leq n-2$, we have

$$\|D^j(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leq c_{jk}^* h^{6-j} \omega(D^6 f; h),$$

where the constants c_{jk}^* are: $c_{0k}^* = \frac{79}{1440}$, $c_{1k}^* = \frac{1}{8}$, $c_{2k}^* = \frac{7}{24}$, $c_{3k}^* = \frac{35}{48}$, $c_{4k}^* = \frac{3}{2}$, $c_{5k}^* = \frac{9}{4}$ and $c_{6k}^* = \frac{3}{2}$.

Proof. Usint the Taylor expansion of $f(x)$, $x_k \leq x \leq x_{k+1}$, and (2.1) we get

$$\begin{aligned} |f(x) - S_k(x)| &\leq \sum_{j=0}^5 \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j + \\ &+ \frac{h^6}{6!} |f^{(6)}(\xi) - S_k^{(6)}| \leq \sum_{j=0}^6 \frac{|f^{(j)}(x_k) - S_k^{(j)}|}{j!} h^j + \\ &+ \frac{h^6}{6!} |f^{(6)}(\xi) - f^{(6)}(x_k)|, \end{aligned}$$

where $x_k < \xi < x_{k+1}$. Using Lemma 3.1, (2.2) and the definition of the modulus of continuity of $f^{(6)}(x)$ we get

$$|f(x) - S_k(x)| \leq \frac{79}{1440} h^6 \omega(D^6 f; h).$$

Similar procedures for the derivatives will easily complete the proof. \square

Finally, we can use the above theorem to prove the following lemma and theorem.

Lemma 3.1. For $k = 0, n-1$ and $j = 1, 2, 3, 5$ and 6 we have

$$|S_k^{(j)} - f^{(j)}(x_k)| \leq c_{j,k} h^{6-j} \omega(D^6 f; h),$$

where the constants $c_{j,k}$ are given by $c_{1,k} = \frac{7}{144}$, $c_{3,k} = \frac{5}{12}$, $c_{5,k} = \frac{5}{4}$ and $c_{6,k} = \frac{5}{2}$.

Theorem 3.2. Let $f \in C^6[x_0, x_n]$ and let S_Δ be the lacunary g -spline given in (2.1)–(2.14). Then for all $0 \leq j \leq 6$ and $k = 0, n-1$ we have

$$\|D^j(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leq c_{j,k}^* h^{6-j} \omega(D^6 f; h),$$

where the constants $c_{j,k}^*$ are given by $c_{0,k}^* = \frac{19}{144}$, $c_{1,k}^* = \frac{95}{288}$, $c_{2,k}^* = \frac{35}{48}$, $c_{3,k}^* = \frac{35}{24}$, $c_{4,k}^* = \frac{5}{2}$, $c_{5,k}^* = \frac{15}{4}$ and $c_{6,k}^* = \frac{5}{2}$.

REFERENCES

- [1] Mishra R. S. and Mathur K. K.: Lacunary interpolation by splines, (0; 0, 1, 4) case. *Acta Math. Acad. Sci. Hungar.* **36** (1980), 251–260.
- [2] Schumaker L.: Spline Functions. Basic Theory. John Wiley and Sons, New York – Toronto, 1981.