MODELLING OF DEPTH FILTRATION

FARKAS M., GARAY B., SZABÓ G., SZÉPKUTI L., V. NAGY I.
Budapest University of Technology, Budapest, H – 1521

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Abstract. A hydraulic-mathematical model in form of a coupled pair of a partial and of an ordinary differential equation describing the clogging process of depth filters is introduced. Existence and uniqueness of solution is shown and a numerical method is proposed.

1. Introduction

As a result of technological development a new type of filter with coarse-granulation porous, flexible, plastic bed was introduced for the purification of waste waters of the transport industry (service stations, factories etc.). In such type of waste waters liquid, solid and oil particles can be found in a form of a special suspension.

In depth filters a large portion of the filter bed is taking part in the filtration process and the efficiency is substantially higher than that of a sand filter working on the surface filtration principle.

For the reliable dimensioning of the filters and the determination of the length of periods of the regeneration (backwashing) one has to characterize the hydraulic process of the depth filtration, i.e. the seepage flow in a saturated porous and compressible medium.

In this paper we are introducing a hydraulic-mathematical model describing the change of the concentration of the suspension flowing through the filter and the clogging process of the filter, in a special case of depth filtration.

In Section 2 the model is set up consisting of a system of partial differential equations. In Section 3 the existence and the uniqueness of the solution of the problem is established. In Section 4 an approximate solution is given and a computer program is presented to implement the results numerically.

2. The description of the model

We assume that the depth filter is of cylindrical (or prismatical) form and its length is L. Contrary to what is usual, we do not measure length along the filter in back current. In what follows the zero point is the one where the waste water enters the filter and the point L is where it is getting out after filtration. This convention makes the handling of the mathematical formalism easier.

The equation of continuity in its simplest form can be stated [1] as

$$\frac{\partial Q}{\partial x} = +\frac{\partial s}{\partial t}, \tag{2.1}$$

where $Q[L^3T^{-1}]$ is the discharge (the sum of the discharges of the liquid and the solid components), $x[L]$ is the length, $s[L^2]$ is the average cross-sectional area of the seepage cores and $t[T]$ is the time. Substituting the values of the specific discharge $q = Q/F[LT^{-1}]$ and of the porosity of the filter $m = s/F$, where $F[L^2]$ is the cross-sectional area of the filter, we obtain

$$\frac{\partial q}{\partial x} = +\frac{\partial m}{\partial t}. \tag{2.2}$$

Let us recall that the porosity of the filter can be expressed as

$$m = m_0(1-\xi)$$

where m_0 is the original porosity (i. e. the ratio of the volume of the original cores to the volume of the filter) and ξ is the colmation porosity (i.e. the specific volume of the settled particles in the process of filtration).

The specific discharge is the sum of the specific discharges of the liquid and the solid components:

$$q = q_l + q_s.$$

We may assume that

$$\frac{\partial q_l}{\partial x} = 0.$$

Consequently, (2.2) goes over into

$$\frac{\partial q_s}{\partial x} = -m_0\frac{\partial \xi}{\partial t}, \tag{2.3}$$

where $\partial\xi/\partial t$ characterizes the process of colmation in time.

The concentration of suspended particles δ can be written as

$$\delta = q_s/q_l,$$

and this, in turn, implies that (2.3) goes over into

$$q_l\frac{\partial \delta}{\partial x} = -m_0\frac{\partial \xi}{\partial t}. \tag{2.4}$$

To solve equation (2.4), it can be presupposed [3] that the clogging process can be characterized by

$$\frac{\partial \xi}{\partial t} = a\frac{\delta}{v}, \tag{2.5}$$

where $v[LT^{-1}]$ is the average seepage velocity.

On the other hand, the relationship between the colmation porosity and seepage velocity can be taken as

(2.6) $$v = \frac{b}{1-\xi}.$$

Introducing the experimental parameter $N = a/b$ $[T^{-1}]$ which will characterize the filtration capacity of the filter and substituting equation (2.6) into equation (2.5) we obtain

(2.7) $$\frac{\partial \xi}{\partial t} = N\delta\,(1-\xi).$$

The initial condition for the colmation porosity is

(2.8) $$\xi(0, x) = 0.$$

The boundary condition for the concentration δ is, naturally, given only at the entrance to the filter

(2.9) $$\delta(t, 0) = \delta_i(t),$$

where $\delta_i(t)$ is the input concentration.

Equations (2.4), (2.7), (2.8), (2.9) form an initial-boundary value problem. On the basis of the solution of this initial-boundary value problem the efficiency of the filtering as well as the necessary time period of regeneration (backwashing) can be determined.

3. Existence and uniqueness of the solution of the initial-boundary value problem

Existence and uniqueness of the solution of problem (2.4), (2.7)–(2.9) follows from a fixed point argument. For this purpose we write the problem in the form of an integral equation.

Integrating (2.7) with respect to t and taking (2.8) into consideration we get

(3.1) $$1-\xi(t,x) = \exp\left(-\int_0^t N\delta(\tau,x)\,d\tau\right)$$

Substituting $\partial\xi/\partial t$ from (2.7) into (2.4) and integrating with respect to x we obtain

(3.2) $$\delta(t,x) = \delta_i(t)\exp\left(-\frac{m_0 N}{q_i(t)}\int_0^x (1-\xi(t,s))\,ds\right),$$

where (2.9) has also been used. Introducing the new variable

(3.3) $$z(t, x) = 1-\xi(t, x)$$

and substituting (3.2) into (3.1) we get the nonlinear integral equation for z:

$$(3.4)\qquad z(t,x) = \exp\left(-\int_0^t N\delta_i(\tau)\exp\left(-\frac{m_0N}{q_l(\tau)}\int_0^x z(\tau,s)\,ds\right)d\tau\right).$$

Since the right hand side of (3.4) is continuously differentiable with respect to t and x, provided that z is continuous, it follows that (3.4) is equvivalent to (2.4), (2.7)–(2.9) indeed.

Let T be a fixed positive constant and $Z = C([O, T]\times[O, L], \mathbf{R})$ be the Banach space of continuous functions mapping the rectangle $[O, T]\times[O, L]$ into $\mathbf{R}$ with the usual maximum norm, i.e. for $z\in Z$

$$\|z\| = \max\{|z(t,x)|\,|t\in[0,T], x\in[0,L]\}.$$

Introducing the operator $F\colon Z\to Z$ by

$$F(z)(t,x) = \exp\left(-\int_0^t N\delta_i(\tau)\exp\left(-\frac{m_0N}{q_l(\tau)}\int_0^x z(\tau,s)\,ds\right)d\tau\right),$$

(3.4) can be considered as an abstract operator equation

$$(3.5)\qquad z = F(z),\quad z\in Z,$$

where F is completely continuous on $0\le\|z\|\le 1$.

By Schauder's fixed point theorem [4] (3.5) has a (not necessarily unique) solution satisfying

$$(3.6)\qquad O<z(t,x)\le 1,\quad O\le t\le T,\quad O\le x\le L.$$

If z_1 and z_2 satisfy (3.6) it is not hard to establish the inequality

$$(3.7)\qquad \|F^n(z_1)-F^n(z_2)\|\le\frac{K^nL^nT^n}{(n!)^2}\|z_1-z_2\|$$

where

$$K = \max\frac{N^2m_0\delta_i(t)}{q_l(t)},\quad 0\le t\le T.$$

Thus, for a sufficiently large n the operator $F^n : Z\to Z$ is a contraction. Hence, by Banach's contraction principle [4] (3.5) has exactly one solution z^* and the sequence of successive approximations $z_0 = 1$, $z_1 = F(z_0),\ldots, z_{n+1} = F(z_n)$, converges uniformly to z^*. For arbitrary n, inequality (3.7) implies

$$\|z_n-z^*\|\le\frac{K^nL^nT^n}{(n!)^2},$$

which tends to zero as n tends to infinity.

Since T can be chosen arbitrarily, we have arrived at the following

Theorem. *The initial-boundary value problem* (2.4), (2.7)–(2.9) *has a unique solution* $\xi^*(t,x)$, $\delta^*(t,x)$ for $O\le t<\infty$, $O\le x\le L$.

It is to be noted that the solution satisfies the inequalitites for $0 \leq t < \infty$, $0 \leq x \leq L$

$$(3.8) \qquad \begin{matrix} \text{(i)}\ 0 \leq \xi^*(t,x) < 1, & \text{(ii)}\ 0 \leq \delta^*(t,x) \leq \delta_i(t) \\ \text{(iii)}\ \partial\xi^*/\partial t \geq 0, & \text{(iv)}\ \partial\delta^*/\partial x \leq 0. \end{matrix}$$

In fact (i) is a simple consequence of (3.6) and of (3.3), (ii) follows from (3.2) and (iii) and (iv) are consequences of the original differential equations.

Inequalities (3.8) are important from the point of view of the validation of our model. As a matter of fact if they did not hold, our model would not be realistic. It is to be expected that the value of the colmation function is between zero and one (i), the concentration is non-negative and not higher than the input concentration (ii), the colmation of the filter is increasing in time (iii) and the concentration of the suspension is decreasing as it is advancing along the length of the filter (iv). Clearly, the validity of our model can be judged only by comparing the values of our solution ξ^*, δ^* to the experimental data.

4. The approximation of the solution

Applying $\exp y \approx 1+y$, $|y| \ll 1$ in the linearization of (3.4) we get

$$z(t,x) \approx 1 - \int_0^t N\delta_i(\tau) \exp\left(-\frac{m_0 N}{q_l(\tau)} \int_0^x z(\tau,s)\,ds\right) d\tau \approx$$

$$\approx 1 - \int_0^t N\delta_i(\tau)\left[1 - \frac{m_0 N}{q_l(\tau)} \int_0^x z(\tau,s)\,ds\right] d\tau.$$

From here it follows that the solution of (3.4) z^* is approximately equal to the solution of the linear operator equation

$$(4.1) \qquad z = a + Az,\ z \in Z$$

where $a \in Z$ is defined by

$$a(t,x) = 1 - N \int_0^t \delta_i(\tau)\,d\tau$$

and the linear operator $A: Z \to Z$ is defined by

$$A(z)(t,x) = \int_0^t \frac{N^2 m_0 \delta_i(\tau)}{q_l(\tau)} \int_0^x z(\tau,s)\,ds\,d\tau.$$

Arguing similarly as in the proof of inequality (3.7), we get

$$\|A^n z\| \leq \frac{K^n L^n T^n}{(n!)^2} \|z\|,\ z \in Z.$$

As a concequence, we get that the series

$$\tilde{z}^* = \sum_{n=0}^{\infty} A^n a \tag{4.2}$$

is convergent and $\tilde{z}^*$ is the unique solution of (4.1). By mathematical induction for n the n-th iterate $A^n a$ can be expressed explicitly:

$$(A^n a)(t,x) = \frac{1}{(n!)^2}(xI(t,0))^n - \frac{1}{(n!)^2}\int_0^t N\delta_l(\tau)(xI(t,\tau))^n\,d\tau$$

where

$$I(t,\tau) = \int_\tau^t \frac{N^2 m_0 \delta_l(s)}{q_l(s)}\,ds.$$

Linearizing (3.2) and substituting $\tilde{z}^*$ in place of z we obtain the approximate solution for the concentration δ in the form

$$\tilde{\delta}^*(t,x) = \delta_l(t)\left[1 - \frac{m_0 N x}{q_l(t)}\Phi(xI(t,0)) + \right.$$

$$\left. + \frac{m_0 N x}{q_l(t)}\int_0^t N\delta_l(\tau)\Phi(xI(t,\tau))\,d\tau\right] \tag{4.3}$$

where

$$\Phi(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!(k+1)!}.$$

By the formulae (4.2) and (4.3) the approximate solution of the problem has been reduced to a process of iterated integration of, in principle, known functions. Thus, if the average specific discharge or seepage velocity of the suspension, the input concentration and the constant parameters characterizing the filter and the liquid are known from measurement or simulation, we can approximately describe the variation of the clogging process and the concentration in time and along the filter. The implementation of these results on computer is relatively easy.

An abstract program of the calculations is presented in the following:

```
PROCEDURE δ̃(t, x) seq
    integral = Ø, τ = t
    CALCULATION OF INTEGRAL iter until τ = Ø
        I = Ø,  s = τ
        CALCULATION OF I(t. τ) iter until s = t
            I = I+(N²m₀δₗ(s)/qₗ(s))·ds
            s = s+ds
        CALCULATION OF I(t, τ) end
        u = x·I, Φ = 1,  k = n
```

```
        CALCULATION OF Φ(x·I(t,τ)) iter until k = 1
            Φ = Φ·u/k(k+1)+1
            k = k-1
        CALCULATION OF Φ(x·I(t, τ)) end
        integral = integral+Nδ_I(τ)·Φ·dτ
        τ = τ-dτ
    CALCULATION OF INTEGRAL end
    δ̃ = δ_I(t)·(1+m_0Nx/q_I(t)·(-Φ+integral))
PROCEDURE δ̃(t, x) end
```

Remark: parameters *ds*, *dτ* (scales of time) and *n* (index of partial sum in series *Φ*(*u*)) are given outside of the procedure, and *t*, *x* are input data for the procedure too.

A simple BASIC version of the above procedure is presented in the following:

```
10  REM DETERMINATION OF CONCENTRATION AT (T, X)
20  INTEGRAL = Ø
30  FOR TA = T TO Ø STEP-DTA
40  I = Ø: FOR S = TA TO T STEP DS:I = I+FND(S)/FNQ(S):
    NEXT S
50  I = I*DS*M*N*N
60  U=X*I:F=1:FOR K=N TO1 STEP-1:F=F*U/K/(K+1)+1:
    NEXT K
70  INTEGRAL=INTEGRAL+FND(T)*N*F
80  NEXT TA
90  INTEGRAL=INTEGRAL*DTA
100 CONCENTRATION=FND(T)*(1+M*N*X/FNQ(T)
    *(-F+INTEGRAL))
```

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