ON DIFFERENTIABLE ADDITIVE FUNCTIONS

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1. Let Λ denote the set of all those real sequences $\{\lambda_n\}$ for which $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $L := \sum_{n=1}^{\infty} \lambda_n < \infty$.

Definition 1.1. A sequence $\{\lambda_n\} \in \Lambda$ is said to be *interval filling*, if for any number $x \in [0, L]$ there exists a sequence $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$, such that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \quad \square \tag{1.1}$$

As is known ([1]), we have the following

Theorem 1.2. *A sequence* $\{\lambda_n) \in \Lambda$ *is interval filling if and only if for any* $n \in \mathbf{N}$

$$\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i \tag{1.2}$$

holds.

Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence and $x \in [0, L]$. Moreover, let us define by induction on n

$$\varepsilon_n(x) := \begin{cases} 1 \text{ if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leqq x \\ 0 \text{ if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x. \end{cases} \tag{1.3}$$

Then ([1], [2])

$$x = \sum_{n-1}^{\infty} \varepsilon_n(x)\lambda_n \tag{1.4}$$

and the representation (1.4) will be called the *regular expansion* of x with respect to the sequence $\{\lambda_n\}$.

Definition 1.3. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call a bounded function $F:[0, L]\to\mathbf{R}$ *additive*, if for any $x\in[0, L]$ the equality

$$(1.5)\qquad F(x) = \sum_{n=1}^{\infty} \varepsilon_n(x)F(\lambda_n)$$

holds, where $\varepsilon_n(x)$ equals 0 or 1 according to the algorithm (1.3). □

It is known that if we put

$$a_n := F(\lambda_n)\ (n\in N),\ \text{then}\ \sum_{n=1}^{\infty} |a_n| < \infty,$$

and conversely any sequence $a_n\in\mathbf{R}$ satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$ determines an additive function.

In this paper we investigate the following question: What can be said about the structure of a differentiable additive function defined on the interval $[0, L]$?

2. Let $\{\lambda_n\}\in\Lambda$ be a fixed interval filling sequence.

Definition 2.1. We call *a* number $x\in[0, L]$ terminating, if there exists a number N, such that $\varepsilon_n(x) = 0$ for $n>N$. If x is terminating and $\varepsilon_m(x) = 1$, moreover $\varepsilon_n(x) = 0$ for $n>m$ then we say that x has *length* m, written $h(x) = m$. □

Lemma 2.1. *Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $F:[0, L]\to\mathbf{R}$ an additive function (with respect to $\{\lambda_n\}$), differentiable at the terminating point $x\in[O, L]$. If $a_n := F(\lambda_n)$ $(n\in\mathbf{N})$ then*

$$(2.1)\qquad \lim_{n\to\infty} \frac{a_n}{\lambda_n} = F'(x).$$

Proof. Let $h(x) = m$, i. e.

$$x = \sum_{i=1}^{m} \varepsilon_i(x)\lambda_i.$$

Then there exists an $N>m$, such that for $n>N$ the relation

$$\varepsilon_k(x+\lambda_n) = \begin{cases} \varepsilon_k(x) & \text{for } k = 1, 2, \dots, m \\ 1 & \text{for } k = n \\ 0 & \text{for } k\in\mathbf{N}\setminus\{1, 2, \dots, m\}\cup\{n\}. \end{cases}$$

Thus

$$\frac{a_n}{\lambda_n} = \frac{F(x+\lambda_n)-F(x)}{\lambda_n}\quad (n>N)$$

and, by differentiability at x, this implies (2.1). □

3. Let us now ask the following more difficult question. What will the consequences be in case the additive function F is differentiable at a *nonterminating* point x? For arbitrary interval filling sequences this question still remains open, nevertheless we are able to give an answer for quite a large class of sequences.

Definition 3.1. We call the interval filling sequence $\{\lambda_n\}\in\Lambda$ *smooth,* if there exists a constant $K>1$, such that

$$(3.1)\qquad \sum_{i=n+1}^{\infty} \lambda_i < K\lambda_n$$

for any $n\in\mathbf{N}$. □

The inequality (3.1) expressing smoothness can be regarded as being complementary to the inequality (1.2).

Theorem 3.2. *Let $\{\lambda_n\}\in\Lambda$ be a smooth interval filling sequence and F: $[0, L]\to\mathbf{R}$ an additive function (with respect to $\{\lambda_n\}$). If F is differentiable at the nonterminating point $x\in[0, L]$, then with the notation $a_n := F(\lambda_n)$ ($n\in\mathbf{N}$) we have*

$$(3.2)\qquad \lim_{\substack{N\to\infty \\ N\in\mathbf{N}_1}} \frac{a_N}{\lambda_N} = F'(x),$$

where $\mathbf{N}_1 := \{n\,|\,n\in\mathbf{N},\ \varepsilon_n(x) = 1\}$.

Proof. Let

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n,$$

where $\mathbf{N}_1 := \{n\,|\,n\in\mathbf{N}, \varepsilon_n(x) = 1\}$ is an infinite index set. Let moreover $S_N(x) := \sum_{n=1}^{N} \varepsilon_n(x)\lambda_n$, where the right-hand side is the regular expansion of $S_N(x)$. Since F is differentiable at x, there exists a function E_x: $]0, L[\to\mathbf{R}$, such that

$$\lim_{y\to x} E_x(y) = 0$$

and

$$(3.3)\qquad F(x)-F(y) = F'(x)(x-y)+E_x(y)(x-y)$$

is satisfied for every $y\in\,]0, L[$.

If $N\in\mathbf{N}_1$ (i.e. $\varepsilon_N(x) = 1$), then (3.3) implies

$$F(x)-F(s_N(x)) = F'(x)(x-s_N(x))+E_x(s_N(x))(x-s_N(x))$$

and

$$F(x)-F(s_{N-1}(x)) = F'(x)(x-s_{N-1}(x))+E_x(s_{N-1}(x))(x-s_{N-1}(x)).$$

Hence

$$a_N = F(s_N(x))-F(s_{N-1}(x)) = F'(x)\lambda_N+E_x(s_{N-1}(x))\left[\lambda_N+\right.$$

$$\left.+\sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i\right]-E_x(s_N(x))\left[\sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i\right].$$

From the previous equality we get for $N\in\mathbf{N}_1$ that

$$\frac{a_N}{\lambda_N} = F'(x)+E_x(s_{N-1}(x))\left[1+\eta_N(x)\right]-E_x(s_N(x))\eta_N(x),$$

where

$$0<\eta_N(x) := \frac{\sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i}{\lambda_N} \leqq \frac{\sum_{i=N+1}^{\infty}\lambda_i}{\lambda_N} < K.$$

Thus for $N\to\infty$ $(N\in\mathbf{N}_1)$ we obtain (3.2). □

4. On the basis of the foregoing we are able now to formulate the following

Theorem 4.1. *Let $\{\lambda_n\}\in\Lambda$ be a smooth interval filling sequence. Let moreover F: $[0, L]\to\mathbf{R}$ be an additive function (with respect to $\{\lambda_n\}$), differentiable at every point. Then there exists a $c\in\mathbf{R}$ such that $F(x) = cx$ for every $x\in[0, L]$.*

Proof. By Lemma 2.1. there exists $c := \lim_{n\to\infty}\frac{a_n}{\lambda_n}$ with $a_n := F(\lambda_n)$. On the basis of Lemma 2.1. and Theorem 3.2. $F'(x) = c$ for any $x\in[0, L]$. Hence $F(x) = cx$. □

It is known that for $1<q<2$ the sequence

$$\lambda_n := \frac{1}{q^n}\ (n\in\mathbf{N})\ \text{is interval filling}\left(L=\frac{1}{q-1}\right).$$

On the other hand, in view of

$$\sum_{i=n+1}^{\infty}\frac{1}{q^i} = \frac{L}{q^n},$$

this sequence is smooth, hence Theorem 4.1. can be applied.

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