RIEMANN BOUNDARY VALUE PROBLEM AND SINGULAR INTEGRAL EQUATIONS ON A NEW CLASS OF FUNCTIONS

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Abstract. In this paper a solution of a Riemann boundary value problem is studied. It is shown that under certain conditions the problem possesses a solution in the class Z_w.

Salaev [7] obtained necessary and sufficient conditions for the continuity of a singular integral with continuous density where the contour of integration are piece-wise smooth curves without cusp points. Papers [2–6] introduce a new class of functions which are used in the study of Riemann boundary value problem and characteristic singular integral equations on some classes of curves which are more general than class of piece-wise smooth curves without cusp points.

Let γ be a closed Jordan rectifiable curve. The interior of γ is denoted by D^+, and the exterior of γ by D^-. Following Salaev [8], we have,

$$\gamma_\delta(t) = \{y \in \gamma : |y-t| \leq \delta\} \quad ,\delta > 0,$$

$$\Theta_t(\delta) = \operatorname{mes} \gamma_\delta(t) \quad ,\delta > 0,$$

$$\Theta_t(\delta) = \sup_{t \in \gamma} \Theta_t(\delta) \quad ,\delta \geq 0,$$

If $\delta \in (0, d]$ where $d = \operatorname{diam} \gamma = \sup_{t,\tau \in \gamma} |t-\tau|$ then the functions $\Theta(\delta)$ are non decreasing, $\lim_{\delta \to 0} \Theta(\delta) = 0$ and $\Theta(\delta) \geq \delta$.

Consider the class of curves, for which $\Theta(\delta) \sim \delta$ (i.e. there exists a constant c such that $\Theta(\delta) \leq c\delta$).

Salaev V. V. and Tokov A. O. [9] obtained the following result. Let $f \in C_\gamma$ (C_γ is the class of continuous functions defined on γ). Then for the continuity, up to contour, of the integral of Cauchy type

$$\Phi(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t-z} dt,$$

it is necessarry and sufficient that the integral

$$\int\limits_{\gamma/\gamma_\varepsilon(t)} \frac{f(\xi)-f(t)}{\xi-t}\,d\xi$$

is convergent uniformly with respect to $t\in\gamma$ as $\varepsilon\to 0$. This condition is called condition *I*.

Now we reformulate the main results [3, 6] for the class of curves which satisfy the condition $\Theta(\delta)\sim\delta$.

Let us denote by S_γ the set of all functions $f\in S_\gamma$ satisfying condition I. As in [6] let us consider the metric characteristic function of $f\in S_\gamma$;

$$\Omega_f(\delta) = \delta \sup_{\tau\geq\delta} \frac{\Omega_f^1(\delta)}{\tau},$$

$$\Omega_f^1(\delta) = \sup_{\substack{t\in\gamma \\ \varepsilon\leq\delta}} \left| \int\limits_{\gamma_\varepsilon(t)} \frac{f(\xi)-f(t)}{\xi-t}\,d\xi \right|.$$

For $f\in C\gamma$ it was introduced in [1] the characteristic function

$$w_f\delta = \delta \sup_{\tau\geq\delta} \frac{w_f^1(\tau)}{\tau}$$

where

$$w_f^1(\delta) = \sup_{|t_1-t_2|\leq\delta} |f(t_1)-f(t_2)|.$$

Theorem 1 [3]. *Let $\Theta(\delta)\sim\delta, f\in S_\gamma$, Then for the singular integral*

$$(Af)(t) = \tilde{f}(t) = \int\limits_{\gamma} \frac{f(\xi)-f(t)}{\xi-t}\,d\xi + \pi i f(t),\; t\in\gamma, \tag{1}$$

the following two inequalities are satisfied

$$w_{\tilde{f}}(\delta)\leq c\left(\Omega_f(\varepsilon)+w_f(\varepsilon)+\delta\int\limits_{\varepsilon}^{\delta}\frac{w_f(\tau)}{\tau^2}\,d\tau\right),$$

and

$$\Omega_{\tilde{f}}(\delta)\leq c\left(\delta\int\limits_{\delta}^{d}\frac{\Omega_f(\xi)}{\xi^2}\,d\xi+\delta\int\limits_{\delta}^{d}\left(\frac{1}{\xi}\int\limits_{\xi}^{\eta}\frac{w_f(\eta)}{\eta}\,d\eta\right)d\xi\right),$$

for any $\varepsilon\in(0, d]$, and for $c>0$ depending only on γ.

Let w be any function of continuity modules type. We introduce a class of functions

$$Z_w = \{f\in S\colon w_f(\delta) = O(w(\delta)),\ \Omega_f(\delta) = O(w(\delta))\}.$$

with norm

$$\|f\|_{Z_w} = \|f\|_{C_\gamma} + \sup_{\delta>0} \frac{w_f(\delta)}{w(\delta)} + \sup_{0>\delta} \frac{\Omega_f(\delta)}{w(\delta)}.$$

It is clear that Z_w is a B-space.

Similarly the Plemeti-Prevolov theorem has been obtained for the class Z_w in the following form.

Theorem 2 [3]. *Let w be any function such that*

$$\delta \int_\delta^d \frac{w(\xi)}{\xi^2} d\xi = O(w(\delta)).$$

Then the operator $Af = \tilde{f}$ maps Z_w to Z_w and is bounded.

Now we reformulate the results of papers [4, 5], to solve the Riemann boundary value problem.

Theorem 3 [5]. *Let $\Theta(\delta) \sim \delta$, $G, g: \gamma \to C$, $G(t) \neq 0$, $\forall\, t \in \gamma$,*

$$\int_0^d \frac{w_G(\xi)}{\xi} \operatorname{Ln} \frac{2d}{\xi} d\xi < +\infty,\ g \in S.$$

Then the general solution of the Riemann boundary value problem, for the determination of a piece-wise holomorphic functions $\Phi(z)$, which tends to zero at infinity under the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),\ t \in \gamma \tag{2}$$

has the form

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\gamma \frac{g(\xi)}{X^+(\xi)(\xi - t)} d\xi + X(z) P_{x-1}(z),$$

where

$$X(z) = \begin{cases} e^{\Gamma(z)}, & z \in D^+, \\ z^{-x} e^{\Gamma(z)}, & z \in D^-, \end{cases}$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{\operatorname{Ln}(\tau^{-x} G(\tau))}{\tau - z} d\tau,$$

P_{x-1} is a polynomial of degree not higher than $x-1$, and x is the index of the problem defined by

$$x = \operatorname{ind}_\gamma G = \frac{1}{2\pi} \int_\gamma d \operatorname{Ln} G(\xi). \tag{3}$$

We introduce here a class of functions

$$H_w = \{f \in C_\gamma : w_f(\delta) = 0(w(\delta))\}$$

with the norm

$$\|f\|_{H_w} = \|f\|_{C_\gamma} + \sup_{\delta>0} \frac{w_f(\delta)}{w(\delta)}.$$

It is clear that H_w is B-space.

Theorem 4. *Let* $\Theta(\delta) \sim \delta$, $a(t)$, $b(t) \in H_\lambda$, $a(t)\big(a(t)-2b(t)\big) \neq 0$, $\forall\, t \in \gamma, f \in Z_w$, *where*

$$\int_0^\delta \frac{\lambda(\xi)}{\xi} \operatorname{Ln} \frac{2d}{\xi} d\xi = O(w(\delta)), \quad \delta \int_\delta^d \frac{\lambda(\xi)}{\xi^2} d\xi = O(w(\delta)),$$

$$\delta \int_\delta^d \frac{w(\xi)}{\xi^2} d\xi = O(w(\delta)).$$

Then for $x \geq 0$ *the general solution of the dominant equation*

$$a(t)\Phi(t) + \frac{b(t)}{\pi i} \int_\gamma \frac{\Phi(\xi)-\Phi(t)}{\xi - t} d\xi = f(t),\ t \in \gamma \tag{4}$$

is contained in the class Z_w *and has form*

$$\Phi(t) = \frac{f(t)}{a(t)} - \frac{b(t)}{a(t)-2b(t)} X^+(t) P_{x-1}(1) - \tag{5}$$

$$- \frac{b(t)X^+(t)}{a(t)-2b(t)} \cdot \frac{1}{i\pi} \int_\gamma \left[\frac{f(\xi)}{a(\xi)X^+(\xi)} - \frac{f(t)}{a(t)X^+(t)} \right] \frac{d\xi}{\xi - t},$$

where $X(z)$ *and* x *represent the canonical solution and the index as in equation* (2), *with*

$$G(t) = \frac{a(t)-2b(t)}{a(t)} \text{ and } \Phi \equiv 0.$$

For $x < 0$, *the necessary and sufficient condition for solving equation* (4) *is*

$$\int_\gamma \frac{f(\xi)\xi^k}{a(\xi)X^+(\xi)} d\xi = 0,\ k = 0, 1, \ldots, -x-1.$$

If the conditions of solubility are satisfied, then the solution of the equation (4) *is given by formula* (5) *where* $P_{x-1}(t) = 0$.

Proof. Let $\Phi \in Z_w$ be the solution of equation (4).
We introduce the function

$$\Phi(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_\gamma \frac{\Phi(\xi)}{\xi - z} d\xi, \, z \notin \gamma,$$

and by using the Plemela-Sokhotski formulae we have

$$\Phi^+(t) = \frac{1}{2\pi i} \int_\gamma \frac{\Phi(\xi) - \Phi(t)}{\xi - t} d\xi + \Phi(t),$$

$$\Phi^-(t) = \frac{1}{2\pi i} \int_\gamma \frac{\Phi(\xi) - \Phi(t)}{\xi - t} d\xi,$$

by substituting in (4) we obtain

$$a(t)(\Phi^+(t) - \Phi^-(t)) + 2b(t)\Phi^-(t) = f(t),$$

or

(6) $$\Phi^+(t) = \frac{a(t) - 2b(t)}{a(t)} \Phi^-(t) + \frac{f(t)}{a(t)}.$$

Thus, the solution Φ of equation (4) is the solution Φ of the boundary value problem (6). It is easy to see that if Φ is the solution of equation (6), then the function $\Phi(t) = \Phi^+(t) - \Phi^-(t)$ is the solution of equation (4) and is contained in class Z_w. I. e. the solution of equation (4) is equivalent to the solution of boundary value problem (6).

Then for $x \geq 0$ the general solution of problem (6) has the form

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\gamma \frac{f(\xi)}{a(\xi) X^+(\xi)} \cdot \frac{d\xi}{\xi - z} + X(z) Q_{x-1}(z),$$

where X is the canonical solution of the homogeneous problem (6) $\left(i.e.\ X^+(t) = \frac{a(t) - 2b(t)}{a(t)} X^-(t) \right)$, and Q_{x-1} is an arbitrary polynomial of degree not higher than $x - 1$ (for $x = 0$, $P_{x-1}(z) = 0$).

Then the solution Φ of equation (4) takes the form

$$\Phi(t) = \Phi^+(t) - \Phi^-(t) = X^+(t) \left(\frac{1}{2\pi i} \int_\gamma \left(\frac{f(\xi)}{a(\xi) X^+(\xi)} - \frac{f(t)}{a(t) X^+(t)} \right) \frac{d\xi}{\xi - t} + \right.$$

$$\left. + \frac{f(t)}{a(t) X^+(t)} \right) + X^+(t) Q_{x-1}(t) - X^-(t) \cdot \frac{1}{2\pi i} \int_\gamma \left(\frac{f(x)}{a(\xi) X^+(\xi)} - \frac{f(t)}{a(t) X^+(t)} \right).$$

$$\cdot \frac{d\xi}{\xi - t} - X^-(t)\,Q_{x-1}(t) = \frac{X^+(t) - X^-(t)}{2\pi i} \int_\gamma \left(\frac{f(\xi)}{a(\xi)X^+(\xi)} - \right.$$

$$\left. - \frac{f(t)}{a(t)X^+(t)} \right) \frac{d\xi}{\xi - t} + \frac{f(t)}{a(t)} + (X^+(t) - X^-(t))Q_{x-1}(t).$$

Then

$$X^+(t) - X^-(t) = X^+(t)\left(1 - \frac{a(t)}{a(t) - 2b(t)}\right) = \frac{-2b(t)}{a(t) - 2b(t)} X^+(t).$$

Finally

$$\Phi(t) = \frac{f(t)}{a(t)} - \frac{b(t)}{a(t) - 2b(t)} \cdot \frac{X^+(t)}{\pi i} \int_\gamma \left(\frac{f(\xi)}{a(\xi)X^+(\xi)} - \right.$$

$$\left. - \frac{f(t)}{a(t)X^+(t)} \right) \frac{d\xi}{\xi - t} + \frac{b(t)X^+(t)}{a(t) - 2b(t)} P_{x-1}(t),$$

where $P_{x-1}(t) = -2Q_{x-1}(t)$ is an arbitrary polynomial of degree not higher than $x - 1$.

By the same way we can find the result in the case $x < 0$. □

Consider now the equation adjoint to the dominant equation which has the form

$$(7) \qquad (a(t) - 2b(t))\psi(t) - \frac{1}{\pi i} \int_\gamma \frac{b(\xi)\,\psi(\xi) - b(t)\,\psi(t)}{\xi - t} d\xi = q(t), \quad t \in \gamma$$

where $a(t)$, $b(t)$, $q(t)$ satisfy the conditions of Theorem 4.

Theorem 5. *Let the functions $a(t)$, $b(t)$, $q(t)$ satisfy the conditions of Theorem 4. Then for $x \leq 0$, the general solution of equation (7) is contained in the class Z_w and has the form*

$$(8) \qquad \begin{aligned} \psi(t) = \frac{b(t)}{a(t)} \frac{1}{\pi i} \int_\gamma \left(\frac{b(\xi)\,q(\xi)}{(a(\xi) - 2b(\xi))X_1^+(\xi)} - \frac{b(t)\,q(t)}{(a(t) - 2b(t))X_1^+(t)} \right) \frac{d\xi}{\xi - t} + \\ + \frac{b(t)\,q(t)}{a(t) - 2b(t)} + \frac{b(t)}{a(t)} P_{1-x}(t), \end{aligned}$$

where $X_1(z)$ represents the canonical solution of the homogeneous Riemann boundary value problem with the coefficient

$$G(t) = \frac{a(t)}{a(t) - 2b(t)},$$

and P_{1-x} is an arbitrary polynomial of degree not higher than $1 - x$.

For $x>0$, the necessary and sufficient condition for which the solution of equation (5) *belongs to Z_w is:*

$$\int_\gamma \frac{q(\xi)\xi^k}{(a(\xi)-2b(\xi))X_1^+(\xi)}d\xi = 0, k = 0, 1, \dots, x-1. \tag{9}$$

If the conditions of solubility are satisfied, the solution of equation (7) *is given by formula* (8), *where $P_{x-1}(t) = 0$.*

Proof. For $x \leq 0$, let $\Psi(t)$ be the solution of equation (7) from class Z_w. Then [4] the function $b(\xi)\, \Psi(\xi) \in Z_w$ and hence for the function

$$\Psi(z) = \frac{1}{2\pi i}\int_\gamma \frac{b(\xi)\,\Psi(\xi)}{\xi - z}\,d\xi$$

in accordance with the Plemela-Sokhotski formulae we found

$$\Psi^+(t) = \frac{1}{2\pi i}\int_\gamma \frac{b(\xi)\,\Psi(\xi)-b(t)\,\Psi(t)}{\xi - t}d\xi + b(t)\,\Psi(t),$$

$$\Psi^-(t) = \frac{1}{2\pi i}\int_\gamma \frac{b(\xi)\,\Psi(\xi)-b(t)\,\Psi(t)}{\xi - t}d\xi.$$

Then

$$b(t)\,\Psi(t) = \Psi^+(t) - \Psi^-(t). \tag{10}$$

From (7) we have

$$\big(a(t)-2b(t)\big)\,\Psi(t) - 2\Psi^-(t) = q(t). \tag{11}$$

Then we obtain that

$$\Psi(t) = \frac{2\Psi^-(t)}{a(t)-2b(t)} + \frac{q(t)}{a(t)-2b(t)}.$$

Then from (10) we have

$$\frac{2b(t)\Psi^-(t)}{a(t)-2b(t)} + \frac{b(t)\,q(t)}{a(t)-2b(t)} = \Psi^+(t) - \Psi^-(t)$$

or

$$\Psi^+(t) = \Psi^-(t)\left(\frac{2b(t)}{a(t)-2b(t)} + 1\right) + \frac{b(t)\,q(t)}{a(t)-2b(t)} =$$

$$= \frac{a(t)}{a(t)-2b(t)}\Psi^-(t) + \frac{b(t)\,q(t)}{a(t)-2b(t)}.$$

Thus, the solutions Ψ of equation (7) is the solution $\Psi(z)$ of the boundary value problem

$$\Psi^+(t) = \frac{a(t)}{a(t)-2b(t)}\Psi^-(t) + \frac{b(t)\,q(t)}{a(t)-2b(t)}. \tag{12}$$

The converse is also true from the following. Let $\Psi(z)$ be the solution of problem (12). Then

$$\Psi(z) = \frac{X_1(z)}{2\pi i}\int_\gamma \frac{b(\xi)\,q(\xi)}{(a(\xi)-2b(\xi))X_1^+(\xi)}\,\frac{d\xi}{\xi - z} + X_1(z)\,Q_{1-x}(z)$$

It is required only to prove that the function

$$\Psi(t) \stackrel{\text{def}}{=} \Psi^+(t) - \Psi^-(t)$$

belongs to Z_w.
We have (see the proof of Theorem 4)

$$\Psi(t) = \frac{X_1^+(t) - X_1^-(t)}{2\pi i}\int_\gamma \left(\frac{b(\xi)\,q(\xi)}{(a(\xi)-2b(\xi))X_1^+(\xi)} - \right. \tag{13}$$

$$\left. - \frac{b(t)\,q(t)}{(a(t)-2b(t))X_1^+(t)}\right)\frac{d\xi}{\xi - t} + (X_1^+(t) - X_1^-(t))Q_{1-x}(t) + \frac{b(t)\,q(t)}{a(t)-2b(t)}.$$

Then from the conditions of this theorem and from Lemma 1 of [4], we can see that

$$\frac{b(t)\,q(t)}{a(t)-2b(t)} \in z_w.$$

To prove that $(X_1^+(t) - X_1^-(t))\;Q_{1-x}(t) \in Z_w$, it is sufficient to prove that $X_1^+(t) - X_1^-(t) \in Z_w$.
From the definition of the canonical function X_1 we have

$$X_1^+(t) = e^{\Gamma_1^+(t)},$$

where

$$\Gamma_1(z) = \frac{1}{2\pi i}\int_\gamma \frac{\operatorname{Ln}\left(\tau^x\left(\frac{a(\tau)}{a(\tau)-2b(\tau)}\right)\right)}{\tau - z}d\tau = \frac{1}{2\pi i}\int_\gamma \frac{\operatorname{Ln} J(\tau)}{\tau - z}d\tau.$$

Then we make an estimate like Zygmund estimation [8].

$$w_{X_1^+}(\delta) \le w_{\Gamma_1^+}(\delta) \le c\left(\int_0^{\delta_w} \frac{w_{\operatorname{Ln} J}(y)}{y}dy + \delta\int_\delta^d \frac{w_{\operatorname{Ln} J}(y)}{y^2}dy\right)$$

$$\leq c\left(\int_0^\delta \frac{w_J(y)}{y}dy+\delta\int_\delta^d \frac{w_J(y)}{y^2}dy\right).$$

It is easy to see that

$$w_J(\delta)\leq \text{const}\,(w_a(\delta)+w_b(\delta))\leq \text{const}\,\lambda(\delta).$$

Then

$$w_{X_1^+}(\delta)\leq\text{const}\left(\int_0^\delta \frac{\lambda(y)}{y}dy+\delta\int_\delta^\delta \frac{\lambda(y)}{y^2}dy\right).$$

Thus $X_1^+\in H_{\tilde\omega}$, where

$$\tilde\omega(\delta)=\int_0^\delta \frac{\lambda(y)}{y}dy+\delta\int_0^d \frac{\lambda(y)}{y^2}dy.$$

But (see [3]) $H_\mu\subset Z_{\bar\omega}$, if

$$\bar\omega(\delta)=\int_0^\delta \frac{\mu(y)}{y}dy,$$

and since

$$\int_0^\delta\left(\int_0^y \frac{\lambda(\eta)}{\eta}d\eta+y\int_y^d \frac{\lambda(\eta)}{\eta^2}d\eta\right)dy\leq$$

$$\leq c\left[\int_0^\delta \frac{\lambda(y)}{y^2}\ln\frac{2d}{y}dy+\delta\int_\delta^d \frac{\lambda(y)}{y^2}dy\right]\leq c\,w(\delta),$$

then $X_1^+\in Z_w$.

By the same way we can prove that

$$X_1^-\in Z_w.$$

Finally we prove that

$$\frac{X_1^+(t)-X_1^-(t)}{2\pi i}\int_\gamma\left(\frac{b(\xi)\,q(\xi)}{(a(\xi)-2b(\xi))X_1^+(\xi)}-\frac{b(t)\,q(t)}{(a(t)-2b(t))\,X_1^+(t)}\right)\frac{d\xi}{\xi-t}\in Z_w.$$

Since

$$\frac{b(t)\,q(t)}{a(t)-2b(t)}\in Z_w,\quad\text{and}$$

$$X_1^+(t) \in H \int_0^\delta \frac{\lambda(y)}{y} dy + \delta \int_\delta^d \frac{\lambda(y)}{y^2} dy,$$

then (since $X^+(t) \neq 0$ and continuous) we find that

$$\frac{1}{X^+(t)} \in H \int_0^\delta \frac{\lambda(y)}{y} dy + \delta \int_\delta^d \frac{\lambda(y)}{y^2} dy.$$

Therefore, from Lemma 1 [4]

$$\frac{b(t)\, q(t)}{(a(t) - 2b(t)) X^+(t)} \in Z_w.$$

Then from Theorem 2

$$\int_\gamma \left(\frac{b(\xi)\, q(\xi)}{(a(\xi) - 2b(\xi))\, X^+(\xi)} - \frac{b(t)\, q(t)}{(a(t) - 2b(t)) X^+(t)} \right) \frac{d}{\xi - t} =$$

$$= A\left(\frac{bq}{(a-2b) X^+} \right)(t) - \pi i \left(\frac{bq}{(a-2b) X^-} \right)(t) \in Z_w.$$

Thus

$$X_1^+(t) - X_1^-(t) \in H \text{ and } \int_0^\delta \frac{\lambda(y)}{y} dy + \delta \int_\delta^d \frac{\lambda(y)}{y^2} dy,$$

end by applying Lemma 1 [4] we obtain the result. Therefore $\Psi \in Z_w$. From (13) we obtain that

$$X_1^+(t) - X_1^-(t) = \frac{2b(t)}{a(t)} X_1^+(t), \text{ and } \Psi \text{ takes the form (8).}$$

By the same way we can prove the same result when $x > 0$. □

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