# ON SYSTEMS WITH BULK ARRIVAL AND GROUP SERVICE. I

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*(Received February 14, 1985)*

**1.** In [1] for the investigation of characteristics of operating systems the tools of queueing theory are frequently used. Usually these models are connected only with the steady-state behaviour in terms of stationary probability distributions, thus the time-dependent case is not considered in these models. In the present paper we describe a new model which may be used for the investigation of nonstationary characteristics of such systems.

First we formulate the corresponding queueing problem. Let $\xi(t)$, $t\geq 0$ be an inhomogeneous Markov chain with state space $\{0,1,\dots\}$ and transition probabilities for $[t, t+\Delta](\Delta\to 0)$:

$$\left\{P\xrightarrow{[t,\,t+\Delta]}k-c+r\right\} = \delta_{rc}+a_r(t)\Delta+o(\Delta),$$

(1)
$$r\geq 0,\, k\geq c,$$

$$\left\{P\xrightarrow{[t,\,t+\Delta]}r\right\} = \delta_{kr}+b_{kr}(t)\Delta+o(\Delta),$$

$$r\geq 0,\, 0\leq k< c,$$

where $b_{kk}(t)<0$, $a_c(t)<0$, $b_{kr}(t)(k\neq r)$ and $a_r(t)(r\neq c)$ nonnegative continuous functions, for which the relations

$$\sum_{r=0}^{\infty} b_{kr}(t) = 0 \text{ and } \sum_{r=0}^{\infty} a_r(t) = 0$$

hold.

Consider a more general process. Associate with $\xi(t)$ a homogeneous second component $\eta(t)\in\mathbf{R}^d$, and create the process $\zeta(t) = \{\xi(t), \eta(t)\}$. The second component will be additive and so it can describe e.g. the costs connected with the functioning of the system. So we get a Markov process with homogeneous second component. These processes with property

$$P\{\xi(s) = r, \eta(s)\in A\,|\,\xi(t) = k, \eta(t) = x\} =$$
$$= P\{\xi(s) = r, \eta(s)\in A_{-x}\,|\,\xi(t) = k, \eta(t) = 0\},$$

where

$$A_{-x} = \{y \in R^d : x+y \in A\},$$

were introduced and investigated by Ezhov and Skorohod in [4] and [5]. We determine $\eta(t)$ depending on the value of $\xi(t)$ as follows:

$$(2)\quad \begin{array}{llll} 1.\ \text{if } \xi(t-0) = k,\ \xi(t+0) = k & \text{and } k \geq c, & \text{then } \eta(t) = \varepsilon(t); \\ 2.\ \text{if } \xi(t-0) = k,\ \xi(t+0) = r & \text{and } k \geq c, & \text{then } \eta(t) = \varepsilon_{r-k}(t); \\ 3.\ \text{if } \xi(t-0) = k,\ \xi(t+0) = k & \text{and } 0 \leq k < c, & \text{then } \eta(t) = \varepsilon_{kk}(t); \\ 4.\ \text{if } \xi(t-0) = k,\ \xi(t+0) = r & \text{and } 0 \leq k < c, & \text{then } \eta(t) = \varepsilon_{kr}(t); \end{array}$$

where $\varepsilon(t)$, $\varepsilon_j(t)$, $\varepsilon_{kr}(t) \in \mathbf{R}^d$ are homogeneous processes, $\eta(t)$ coincides with one of them, the choice depends on the actual jump of the Markov chain.

Moreover, let

$$M(e^{i(\lambda,\, \varepsilon(s)-\varepsilon(t))} \mid \xi(x) = \text{const}, x \in [t, s]) = \exp\left\{\int_t^s f(x, \lambda)\, dx\right\},$$

$$M(e^{i(\lambda,\, \varepsilon_{r-k}(t+0)-\varepsilon_{r-k}(t-0))} \mid \xi(t+0) - \xi(t-0) = r-k) = \varphi_{r-k}(t, \lambda),$$

$$M(e^{i(\lambda,\, \varepsilon_{kk}(s)-\varepsilon_{kk}(t))} \mid \xi(x) = \text{const}, x \in [t, s]) = \exp\left\{\int_t^s f_k(x, \lambda)\, dx\right\},$$

$$M(e^{i(\lambda,\, \varepsilon_{kr}(t+0)-\varepsilon_{kr}(t-0))} \mid \xi(t-0) = k,\ \xi(t+0) = r) = \varphi_{kr}(t, \lambda),$$

where $f(t, \lambda)$ *and* $f_k(t, \lambda)$ are cumulants of processes with independent increments in $\mathbf{R}^d$, $\varphi_r(t, \lambda)$ and $\varphi_{kr}(t, \lambda)$ are characteristic functions for certain distributions in $\mathbf{R}^d$. It can be easily seen that we get our original system with $\lambda = 0$.

**2.** The described Markov chain with homogeneous second component described above is regular. We derive the direct system of Kolmogorov differential equations for the transition probabilities

$$P_{kr}(t, s, A) = P\{\xi(s) = r,\ \eta(s) - \eta(t) \in A \mid \xi(t) = k\}.$$

Here $A$ is the set of possible values of the second component. Let

$$P_{kr}(t, s, \lambda) = \int_{R^d} e^{i(\lambda, z)} P\{\xi(s) = r, \eta(s) - \eta(t) \in dz \mid \xi(t) = k\}$$

and determine $P_{kr}(t, t+\Delta, \lambda)$:

I. $k \geq c$, $r \geq 0$. 1. $r \neq c$

$$P_{k,\, k-c+r}(t, t+\Delta, \lambda) = [a_r(t)\Delta + o(\Delta)] M e^{i(\lambda,\, \varepsilon_{r-c}(t+\Delta)-\varepsilon_{r-c}(t))} =$$

$$= [a_r(t)\Delta + o(\Delta)]\varphi_{r-c}(t, \lambda) = a_r(t, \lambda)\Delta + o(\Delta);$$

2. $r = c$

$$P_{k,k}(t, t+\Delta, \lambda) = [1 + a_c(t)\Delta + o(\Delta)]Me^{i(\lambda, \varepsilon(t+\Delta)-\varepsilon(t))} =$$

$$= [1 + a_c(t)\Delta + 0(\Delta)] \exp\left\{\int_t^{t+\Delta} f(x, \lambda)\, dx\right\} = [1 + a_c(t)\Delta + o(\Delta)] \cdot$$

$$\cdot [1 + f(t, \lambda)\Delta + 0(\Delta)] = 1 + [a_c(t) + f(t, \lambda)]\Delta + o(\Delta) =$$

$$= 1 + a_c(t, \lambda)\Delta + o(\Delta);$$

II. $0 \leq k < c$. 1. $k \neq r$

$$P_{k,r}(t, t+\Delta, \lambda) = [b_{kr}(t)\Delta + o(\Delta)]Me^{i(\lambda, \varepsilon_{kr}(t+\Delta)-\varepsilon_{kr}(t))} =$$

$$= [b_{kr}(t)\Delta + o(\Delta)]\varphi_{kr}(t, \lambda) = b_{kr}(t, \lambda)\Delta + o(\Delta);$$

2. $k = r$

$$P_{k,k}(t, t+\Delta, \lambda) = [1 + b_{kk}(t)\Delta + o(\Delta)]M_e^{i(\lambda, \varepsilon_{kk}(t+\Delta)-\varepsilon_{kk}(t))} =$$

$$= [1 + b_{kk}(t)\Delta + o(\Delta)]exp\left\{\int_t^{t+\Delta} f_k(x,\lambda)dx\right\} = [1 + b_{kk}(t)\Delta + o(\Delta)] \cdot$$

$$\cdot [1 + f_k(t,\lambda)\Delta + o(\Delta)] = 1 + [b_{kk}(t) + f_k(t,\lambda)]\Delta + o(\Delta) =$$

$$= 1 + b_{kk}(t, \lambda)\Delta + o(\Delta).$$

We have

$$P_{lk}(t, s+\Delta, \lambda) = \sum_{i=0}^{c-1} P_{li}(t, s, \lambda)[\delta_{ik} + b_{ik}(s, \lambda)\Delta + o(\Delta)] + \tag{3}$$

$$+ \sum_{i=c}^{k+c} P_{li}(t, s, \lambda)[\delta_{ik} + a_{k-i+c}(s, \lambda)\Delta + o(\Delta)],\ l \geq 0.$$

To establish (3) it is enough to present it in case $0 \leq k < c$ in the form

$$P_{lk}(t, s+\Delta, A) =$$

$$= \sum_{\substack{i=0 \\ i \neq k}}^{c-1} \int_{R^d} P_{li}(t, s, dx)[b_{ik}(s)\Delta + o(\Delta)]P\{\varepsilon_{ik}(s+\Delta) - \varepsilon_{ik}(s) + x \in A\} +$$

$$+ \int_{R^d} P_{lk}(t, s, dx)[1 + b_{kk}(s)\Delta + o(\Delta)]P\{\varepsilon_{kk}(s+\Delta) - \varepsilon_{kk}(s) + x \in A\} +$$

$$+ \sum_{i=c}^{k+c} \int_{R^d} P_{li}(t, s, dx)[a_{k-i+c}(s)\Delta + o(\Delta)]P\{\varepsilon_{k-i}(s+\Delta) - \varepsilon_{k-i}(s) + x \in A\},$$

and in case $k \geq c$

$$P_{lk}(t, s+\Delta, A) =$$

$$= \sum_{i=0}^{c-1} \int_{R^d} P_{li}(t, s, dx)[b_{ik}(s)\Delta + o(\Delta)]P\{\varepsilon_{ik}(s+\Delta) - \varepsilon_{ik}(s) + x \in A\} +$$

$$+ \sum_{\substack{i=c \\ i \neq k}}^{k+c} \int_{R^d} P_{li}(t, s, dx)[a_{k-i+c}(s)\Delta + o(\Delta)]P\{\varepsilon_{k-i}(s+\Delta) - \varepsilon_{k-i}(s) + x \in A\} +$$

$$+ \int_{R^d} P_{lk}(t, s, dx)[1 + a_c(s)\Delta + o(\Delta)]P\{\varepsilon(s+\Delta) - \varepsilon(s) + x \in A\},$$

and to apply the Fourier transform. From (3)

(4)
$$\frac{\partial P_{lk}(t, s, \lambda)}{\partial s} = \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) b_{ik}(s, \lambda) + \sum_{i=c}^{k+c} P_{li}(t, s, \lambda) a_{k-i+c}(s, \lambda) =$$
$$= \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) b_{ik}(s, \lambda) + \sum_{i=0}^{k} P_{l,\, i+c}(t, s, \lambda) a_{k-i}(s, \lambda).$$

Introduce the notations

(5)
$$b_i(t, \lambda, \Theta) = \frac{1}{\Theta^i} \sum_{k=0}^{\infty} b_{ik}(t, \lambda)\, \Theta^k, \ i = 0, 1, \dots, c-1;$$
$$a(t, \lambda, \Theta) = \frac{1}{\Theta^c} \sum_{k=0}^{\infty} a_k(t, \lambda) \Theta^k,$$
$$P_l(t, s, \lambda, \Theta) = \sum_{k=0}^{\infty} P_{lk}(t, s, \lambda) \Theta^k, \ l \geq 0.$$

Multiplying both sides of (4) by $\Theta^k$ and summing from 0 till $\infty$ we get

(6)
$$\frac{\partial P_l(t, s, \lambda, \Theta)}{\partial s} = P_l(t, s, \lambda, \Theta) a(s, \lambda, \Theta) +$$
$$+ \sum_{i=0}^{c-1} P_{li}(t, s, \lambda) \Theta^i [b_i(s, \lambda, \Theta) - a(s, \lambda, \Theta)]$$

with initial condition

(7)
$$P_l(t, t, \lambda, \Theta) = \Theta^l.$$

So we proved

**Theorem 1.** *If*

$$P_{lk}(t, s, \lambda) = \int_{R^d} e^{i(\lambda, z)} P\{\xi(s) = k, \eta(s) - \eta(t) \in dz \,|\, \xi(t) = l\},$$

*where $P\{\xi(s) = k \,|\, \xi(t) = l\}$ are the transition probabilities for inhomogeneous Markov chain $\xi(t)$ with local characteristics* (1), *the second component is deter-*

*mined by* (2), *then the generating function of transition probabilities* $P_l(t, s, \lambda, \Theta)$ *is the solution of equation* (6) *with initial condition* (7) *for any fixed* $l \geq 0$, *where* $a(t, \lambda, \Theta)$ *and* $b_i(t, \lambda, \Theta)$ $(i = 0, 1, \ldots, c-1)$ *are determined by* (5).

**3.** (6) is a linear differential equation for $P_l(t, s, \lambda, \Theta)$, so its solution according to [6] is

$$P_l(t,s,\lambda,\Theta) = \exp\left\{\int_t^s a(x,\lambda,\Theta)dx\right\}\Bigg[\Theta^l + \sum_{i=0}^{c-1}\int_t^s P_{li}(t,y,\lambda)\Theta^i[b_i(y,\lambda,\Theta) -$$

$$(8) \quad -a(y,\lambda,\Theta)]\exp\left\{-\int_t^y a(x,\lambda,\Theta)dx\right\}dy\Bigg] = \exp\left\{\int_t^s a(x,\lambda,\Theta)\,dx\right\}\Theta^l +$$

$$+\sum_{i=0}^{c-1}\int_t^s P_{li}(t,y,\lambda)\Theta^i[b_i(y,\lambda,\Theta) - a(y,\lambda,\Theta)]\exp\left\{\int_y^s a(x,\lambda,\Theta)\,dx\right\}dy.$$

We introduce the auxiliary process with independent increments $\xi^*(t)$ with state space $\{0, \pm 1, \pm 2, \ldots\}$ and transition probabilities for $[t, t+\Delta]$

$$P\{k \xrightarrow{[t,t+\Delta]} k-c+r\} = \delta_{cr} + a_r(t)\Delta + o(\Delta).$$

Then

$$M\Theta^{\xi^*(s)-\xi^*(t)}e^{i(\lambda,\,\eta(s)-\eta(t))} = \exp\left\{\int_t^s a(x,\lambda,\Theta)\,dx\right\}$$

Let further

$$(9) \qquad \varrho_k(t,s,\lambda) = \frac{1}{2\pi i}\oint_{|\Theta|=1}\exp\left\{\int_t^s a(x,\lambda,\Theta)dx\right\}\frac{d\Theta}{\Theta^{k+1}}.$$

Now comparing the coefficients of $\Theta^k$ in (8) we obtain

$$P_{lk}(t,s,\lambda) = \varrho_{k-l}(t,s,\lambda) +$$

$$+\sum_{i=0}^{c-1}\int_s^t P_{li}(t,y,\lambda)\sum_r[b_{ir}(y,\lambda) - a_r(y,\lambda)]\varrho_{k-r-i}(y,s,\lambda)\,dy.$$

Since

$$\exp\left\{\int_t^s a(x,\lambda,\Theta)\,dx\right\} = \sum_{j=-\infty}^{\infty}\varrho_j(t,s,\lambda)\Theta^j,$$

we get

$$-a(y,\lambda,\Theta)\exp\left\{\int_y^s a(x,\lambda,\Theta)\,dx\right\} = \sum_{j=-\infty}^{\infty}\frac{\partial\varrho_j(y,s,\lambda)}{\partial y}\Theta^j,$$

and

$$\sum_r [b_{ir}(y,\lambda)-a_r(y,\lambda)]\varrho_{k-r-i}(y,s,\lambda) =$$
$$= \sum_r b_{ir}(y,\lambda)\varrho_{k-r-i}(y,s,\lambda)+\frac{\partial \varrho_{k-i}(y,s,\lambda)}{\partial y}.$$

Thus we have proved

**Theorem 2.** *$P_{lk}(t,s,\lambda)$ is equal to*

(10)
$$P_{lk}(t,s,\lambda) = \varrho_{k-l}(t,s,\lambda)+$$
$$+\sum_{i=0}^{c-1}\int_t^s P_{li}(t,y,\lambda)\left[\sum_r b_{ir}(y,\lambda)\varrho_{k-r-i}(y,s,\lambda)+\frac{\partial \varrho_{k-i}(y,s,\lambda)}{\partial y}\right]dy.$$

*where $\varrho_k(t,s,\lambda)$ are determined by* (9).

**4.** In oder to compute $P_{lk}(t,s,\lambda)$ $(k = 0, 1, \dots)$ it is necessary to determine the unknown $P_{li}(t. s, \lambda)$ $(k = 0, 1, \dots)$ it is necessary to determine the unknown $P_{li}(t,s,\lambda)$ $(i = 0, 1, \dots, c-1)$. We introduce the notations

(11)
$$\pi_{ik}(t,s,\lambda) = \sum_r b_{ir}(t,\lambda)\varrho_{k-r-i}(t,s,\lambda)+\frac{\partial \varrho_{k-i}(t,s,\lambda)}{\partial t},$$
$$F(t,s,\lambda) = \|\pi_{ik}(t,s,\lambda)\|,\ i,k = 0,1,\dots,c-1,$$
$$\vec{P}_l(t,s,\lambda) = \{P_{l0}(t,s,\lambda), P_{l1}(t,s,\lambda),\dots,P_{l,c-1}(t,s,\lambda)\},$$
$$\vec{\varrho}_l(t,s,\lambda) = \{\varrho_{-l}(t,s,\lambda), \varrho_{1-l}(t,s,\lambda),\dots,\varrho_{c-1-l}(t,s,\lambda)\}.$$

Now from (10) we obtain the system of equations

$$\vec{P}_l(t,s,\lambda) = \vec{\varrho}_l(t,s,\lambda) + \int_t^s \vec{P}_l(t,y,\lambda)F(y,s,\lambda)\,dy.$$

According to the theory of the Volterra integral equations (see e.g. [10]) for any fixed $T$ we get

$$\vec{P}_l(t,s,\lambda) = \vec{\varrho}_l(t,s,\lambda)+\int_t^s \vec{\varrho}_l(t,y,\lambda)G(y,s,\lambda)\,dy\ (0\le s\le t\le T),$$

where $G(t,s,\lambda)$ is the resolvent of the matrix $F(t,s,\lambda)$. Let

(12)
$$\vec{f}_k(t,s,\lambda) = \begin{bmatrix}\pi_{0k}(t,s,\lambda)\\ \pi_{1k}(t,s,\lambda)\\ \dots\dots\dots\dots\\ \pi_{c-1,k}(t,s,\lambda)\end{bmatrix},\ k\ge c,$$

then

$$P_{lk}(t,s,\lambda) = \varrho_{k-l}(t,s,\lambda) + \int_t^s \vec{P}_l(t,y,\lambda)\vec{f}_k(y,s,\lambda)\,dy = \varrho_{k-l}(t,s,\lambda) +$$

(13)

$$+ \int_t^s \vec{\varrho}_l(t,y,\lambda)\vec{f}_k(y,s,\lambda)\,dy + \iint_{t\leq x\leq y\leq s} \vec{\varrho}_l(t,x,\lambda)\, G(x,y,\lambda)\vec{f}_k(y,s,\lambda)\,dxdy,$$

$k \geq c$.

Thus we have proved

**Theorem 3.** *The transition probabilities* $P_{lk}(t, s, \lambda)$ *of* $\zeta(t)$ *are determined by* (13), *where* $\varrho_k(t, s, \lambda)$ *are given by* (9), $\vec{f}_k(t, s, \lambda)$ *by* (11) *and* (12), *and* $G(t, s, \lambda)$ *is the resolvent of matrix* $F(t, s, \lambda)$.

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