THE EQUILIBRIUM PROBLEM FOR A LINEAR MODEL OF OLIGOPOLY WITH MULTI-PRODUCT FIRMS

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Abstract. In this paper we shall be concerned with characterizing the Cournot equilibrium points for a linear oligopoly model with multi-product firms in relation to certain problems in mathematical programming, that is, quadratic programming and linear complementarity problems. The negative definiteness of the coefficient matrix of the inverse demand functions is shown to be of crucial importance in order to guarantee the uniqueness of the equilibrium point.

1. Introduction

In this paper we shall be concerned with characterizing the Cournot equilibrium problem for a linear oligopoly model with multi-product firms for mathematical programming point of view. Theocharis [15] has derived an explicit solution for the linear Cournot oligopoly model without product differentiation, Theocharis' solution has been subject to some unwarranted criticisms until Gehring [6] has derived a complete solution on the basis of a z-transform. An alternative approach to the explicit solution of the linear Cournot model without product differentiation has been proposed by Manas [7] in relation to the solution of a convex programming problem. The Cournot oligopoly game with product differentiation and single product firms has been discussed by several authors (see Friedman [4] and Okuguchi [9]). Little, however, has been done on the Cournot oligopoly problem with multi-product firms, the exceptions being the works by Eichhorn [2, 3], Selten [10] and Bird [1] who have investigated a price-adjusting linear Cournot oligopoly model. Szidarovszky and Okuguchi [13] have derived an existence theorem for the equilibrium points for output-adjusting Cournot oligopoly games with multi-product firms under general nonlinear conditions for the demand and/or cost functions.

Here we are interested in characterizing the Cournot equilibrium problem for an output-adjusting linear oligopoly model with multi-product firms in relation to certain problems in mathematical programming, that is, quadratic programming and linear complementarity problems. The relation between the Cournot equilibrium problem for a non-linear oligopoly model without product differentiation and the solution to a non-linear complementarity problem has been discussed first by Gabay and Moulin [5], and then by Okuguchi [9] who has noted also that the competitive equilibrium for a Walrasian general equilibrium model can be computed as a solution to a non-linear complementarity problem.

2. The cournot equilibrium problem and mathematical programming

Let n be the number of the firms producing m differentiated products. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})$ denote the i-th firm's output vector, where x_{ij} denotes its output of the j-th product. Assume that $S_i \equiv \prod_{j=1}^{m} [0, \bar{\chi}_{ij}]$ is the feasible production set of firm i, where $\bar{\chi}_{ij}$, a positive number, is its capacity limit for the j-th product. We assume that the demand functions are all linear and given as

$$x_j = \sum_{k=1}^{m} \alpha_{jk} p_k + \beta_j, \; j = 1, 2, \ldots, m, \tag{1}$$

where p_k is the price of the k-th product and $X_j \equiv \sum_i x_{ij}$ is the total output of the j-th product produced by all firms. We assume that the elements of the coefficient matrix of (1) satisfy

$$\alpha_{jj} < 0, \; \alpha_{jk} \geq 0, \; j \neq k, \; j, k = 1, 2, \ldots, m, \tag{2}$$

and that

$$|\alpha_{jj}| > \sum_{k \neq j} |\alpha_{jk}|, \; j = 1, 2, \ldots, m, \tag{3a}$$

$$|\alpha_{jj}| > \sum_{k \neq j} |\alpha_{kj}|, \; j = 1, 2, \ldots, m. \tag{3b}$$

The row diagonal dominance assumption (3a) for the Jacobian or coefficient matrix of the demand functions implies that equal increases (or decreases) in the prices of all goods will bring forth a decrease (or increase) in the demand for any product. The column dominance assumption, on the other hand, implies that the own price effect of any product dominates the sum of all its cross effects in the sense that (3b) is satisfied. Since the coefficient matrix of (1) is non-singular under (3a) or (3b), (1) is invertible. Hence we can write

$$p_j = \sum_{k=1}^{m} a_{jk} X_k + d_j, \; j = 1, 2, \ldots, m. \tag{4}$$

Invoking the Heertje theorem (see Selten [10]), we can prove that

(5) $a_{jj}<0,\ a_{jk}\leq 0,\ j\neq k,\ j,\ k = 1, 2, \ldots, m$ and that

(6a) $$|a_{jj}| > \sum_{k\neq j} |a_{jk}|,\ j = 1, 2, \ldots, m,$$

(6b) $$|a_{jj}| > \sum_{k\neq j} |a_{kj}|,\ j = 1, 2, \ldots, m.$$

In view of (5) and (6):

(7) $$2|a_{jj}| > \sum_{k\neq j} |a_{jk}+a_{kj}|,\ j = 1, 2, \ldots, m.$$

Let $\mathcal{A} = [a_{jk}]$ be the coefficient matrix of the inverse demand functions (4). Then (5) coupled with (7) shows that $\mathcal{A}+\mathcal{A}'$ is a matrix with negative dominant diagonals and hence, it is negative definite.

The i-th firm's total cost is assumed to be given as a linear function of its outputs:

(8) $$c_i(\mathbf{x}_i) = \sum_j a_{ij}x_{ij}+c_i,\ i = 1, 2, \ldots, m,$$

where $b_{ij}\geq 0$ for all i and j, and $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})$. The i-th firm's profit function is given as

(9) $$\pi_i = \sum_j x_{ij}p_j - c_i(\mathbf{x}_i),\ i = 1, 2, \ldots, n.$$

We assume that all firms form expectations on other firms' output vectors a la Cournot. It has been shown (see Szidarovszky and Okuguchi [13]) that under the above conditions the function π_i is concave in $\mathbf{x}_i$ for all i, and the non-cooperative game with sets S_i' s of strategies and payoff functions π_i' s has at least one Nash (that is, Cournot) equilibrium point. Hence $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)'$ is the Cournot equilibrium if and only if the following relations are satisfied for each element of x_{ij} of $\mathbf{x}_i$ for all i:

(10) $$\partial\pi_i/\partial x_{ij}\begin{cases}\leq 0 \text{ if } x_{ij} = 0\\ \geq 0 \text{ if } x_{ij} = \bar{\chi}_{ij}\\ = 0 \text{ if } 0<x_{ij}<\bar{\chi}_{ij}.\end{cases}$$

In view of (4), (8) and (9), (10) is equivalent to

(11) $$\sum_k x_{ik}a_{kj} + \sum_k a_{jk}X_k + d_j - b_{ij}\begin{cases}\leq 0 \text{ if } x_{ij} = 0\\ \geq 0 \text{ if } x_{ij} = \bar{\chi}_{ij}\\ = 0 \text{ if } 0<x_{ij}<\bar{\chi}_{ij}.\end{cases}$$

By introducing the slack variables,

$$W_{ij}\equiv\bar{\chi}_{ij}-x_{ij},$$

$$v_{ij}\begin{cases}= 0 \text{ if } x_{ij}<\bar{\chi}_{ij}\\ \geq 0 \text{ otherwise}\end{cases},\quad z_{ij}\begin{cases}= 0 \text{ if } x_{ij}>0\\ \geq 0 \text{ otherwise}\end{cases},$$

relations (11) can be rewritten as

$$\sum_k x_{ik}a_{kj}+\sum_k a_{jk}X_k+d_j-b_{ij}+v_{ij}-z_{ij}=0. \tag{12}$$

Introduce next the following block-matrix:

$$Q=\begin{bmatrix}\mathcal{A}+\mathcal{A}' & \mathcal{A} & \dots & \mathcal{A}\\ \mathcal{A} & \mathcal{A}+\mathcal{A}' & \dots & \mathcal{A}\\ \dots & \dots & \dots & \dots\\ \mathcal{A} & \mathcal{A} & & \mathcal{A}+\mathcal{A}'\end{bmatrix},$$

where $\mathcal{A}=[a_{jk}]$ is the coefficient matrix of (4). Then simple calculation shows that (12) and the definition of the slack variables are equivalent to the following relations:

$$Q\mathbf{x}+\mathbf{d}-\mathbf{b}+\mathbf{v}-\mathbf{z}=\mathbf{0}$$

$$\mathbf{x}+\mathbf{w}=\chi$$

$$\mathbf{x}'\mathbf{z}=\mathbf{v}'\mathbf{w}=\mathbf{v}'\mathbf{z}=0,\ \mathbf{x},\ \mathbf{v},\ \mathbf{z},\ \mathbf{w}\geq\mathbf{0}, \tag{13}$$

where the elements of vectors $\mathbf{x}$, $\mathbf{b}$, $\mathbf{v}$, $\mathbf{z}$, $\mathbf{w}$, $\bar{\chi}$ are x_{ij}, b_{ij}, v_{ij}, z_{ij}, w_{ij}, $\bar{\chi}_{ij}$, respectively, and $\mathbf{d}=(d_1,\dots,d_m,\dots,d_1,\dots,d_m)'$. Hence we have proven the following result.

Theorem 1. *If the demand functions satisfy* (2) *and* (3), *hence* (5) *and* (6), $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ *is the Cournot equilibrium point for the linear oligopoly model with multi-product firms if and only if* (13) *holds.*

Remark. The above derivations imply that this theorem remains true under a more general assumption that $\mathcal{A}+\mathcal{A}'$ is negative definite.

Assume now that the matrix $\mathcal{A}$ is symmetric. Under this additional assumption, (5) and (6a) or (6b) imply that $\mathcal{A}$ is negative definite. In this case the Cournot equilibrium problem is equivalent to a solution of a quadratic programming problem, that is, the following theorem is true.

Theorem 2. *If $\mathcal{A}$ is symmetric and satisfies* (5) *and* (6), *then the Cournot equilibrium point for the linear oligopoly model is an optimal solution of the quadratic programming problem:*

$$\textit{Maximize}\ \mathbf{x}'Q\mathbf{x}+(\mathbf{d}-\mathbf{b})'\mathbf{x}$$

$$\textit{subject to}$$

$$\mathbf{0}\leq\mathbf{x}\leq\bar{\chi}. \tag{14}$$

Moreover the equilibrium point is unique.

Proof. It is known (see Szidarovszky [11]) that the block-matrix Q is negative definite, and in this case the Kuhn-Tucker conditions are necessary and sufficient. The Kuhn-Tucker conditions for problem (14) are relations (13) without the equation $\mathbf{v}'\mathbf{z}=0$. Since the Cournot equilibrium point for the linear model satisfies (13), the Kuhn-Tucker conditions are also satisfied. Hence it is an optimal solution of (14). The uniqueness is ensured by strict concavity of the objective function in (14). □

Remark. The theorem without the uniqueness part holds true if $\mathcal{A}$ is assumed to be symmetric and negative semi-definite.

In order to see the relation between the Cournot equilibrium and a solution to a linear complementarity problem, consider the following problem:

Find non-negative vectors $\mathbf{x}\geq\mathbf{0}$, $\mathbf{v}\geq\mathbf{0}$, $\mathbf{z}\geq\mathbf{0}$, $\mathbf{w}\geq\mathbf{0}$ such that

$$\begin{bmatrix} Q & I \\ -I & O \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}+\begin{bmatrix} \mathbf{d}-\mathbf{b} \\ \bar{\chi} \end{bmatrix}=\begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}$$

$$\mathbf{x}'\mathbf{z} = 0, \ \mathbf{v}'\mathbf{w} = 0. \tag{15}$$

By observing that the first and second equalities in (13) can be rewritten as the first constraint in (15), we have the following result.

Theorem 3. *Under the conditions of Theorem 1, the Cournot equilibrium point, that is, the solution of* (13) *is a solution of the linear complementary problem* (15). *If the solution of* (15) *satisfies the additional constraint* $\mathbf{v}'\mathbf{z} = 0$, *then it is a solution of* (13). *Moreover, the Cournot equilibrium point is unique if problem* (15) *has a unique solution.*

Assume, finally, that the capacity constraints are nonbinding for all firms and for all of the products. In this case v_{ij} in (13) satisfies $v_{ij} = 0$ for all i and j, and the condition $\mathbf{x}+\mathbf{w} = \bar{\chi}$ vanishes. Hence the following result is true.

Theorem 4. *If the same conditions as for Theorem 1 are satisfied, and if, in addition, the capacity constraints are non-binding for all firms and for all of the products, the Cournot equilibrium point for the linear oligopoly modell with multi-product firms and the solution of the following linear complementary problem are identical:*

Find $\mathbf{x}\geq\mathbf{0}$, $\mathbf{z}\geq\mathbf{0}$ *such that*

$$Q\mathbf{x}+\mathbf{d}-\mathbf{b} = \mathbf{z}$$

$$\mathbf{x}'\mathbf{z} = 0. \tag{16}$$

3. Concluding remarks

We have characterized the Cournot equilibrium problem for an oligopoly game with multiproduct firms with linear demand and cost functions as the $\mathbf{x}$-part of a solution of a system of equations (13). We have also shown the relation between the Cournot equilibrium problem and a solution of a certain quadratic programming problem, and that between the Cournot equilibrium problem and a solution to a linear complementarity problem. If the capacity constraints are nonbinding, the Cournot equilibrium point is identical to a solution to the special linear complementarity problem (16). Hence any available algorithm for finding a solution of concave quadratic programming problems or linear complementarity problems can be applied to compute Cournot equilibrium points. For practical methods see e.g. Szidarovszky and Yakovitz [14].

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