ON THE EXISTENCE AND COMPUTATION OF EQUILIBRIUM POINTS FOR AN OLIGOPOLY GAME WITH MULTI-PRODUCT FIRMS

KOJI OKUGUCHI* and FERENC SZIDAROVSZKY**

* Dept. of Economics, Metropolitan University,
Tokyo, Yakumo, Meguro-Ku

** Inst. of Math. and Comp. Sci., K. Marx University of Economics
H – 1093 Budapest, Dimitrov tér 8.

(Received January 14, 1985)

Abstract. In this paper the existence of the Cournot-Nash (or Cournot or Nash) equilibrium and numerical methods for its computation are investigated for an oligopoly with multi-product firms. First the relations between the Cournot-Nash equilibrium points of the N-person noncooperative games and the fixed points of point-to-set mappings are discussed. A sufficient condition is then given for the existence of the equilibrium for a Cournot oligopoly with multi-product firms, based on a new result on concave functions. The condition is interpreted especially in relation to the properties of the demand functions.

1. Introduction

The purpose of this paper is to present some new results on a Cournot oligopoly with multi-product firms which is an N-person noncooperative game with players or firms 1, 2, ..., N. The number of products equals M and the strategy of each player or firm is characterized by the production vector

$$\mathbf{x}_k = (x_k^{(1)}, \ldots, x_k^{(M)})', \tag{1}$$

where $x_k^{(m)}$ is the k-th firm's output of the m-th product. The total production level vector for the oligopoly comprising of N firms is defined as

$$\mathbf{s} \equiv \sum_{k=1}^{N} \mathbf{x}_k. \tag{2}$$

The set of strategies of player k $(1 \le k \le N)$ is then given by the subset S_k of the M-dimensional Euclidean space. The payoff or profit function of firm k is given as

$$\varphi_k(\mathbf{x}_1, \ldots, \mathbf{x}_N) \equiv \mathbf{x}_k'\mathbf{f}(\mathbf{s}) - C_k(\mathbf{x}_k), \tag{3}$$

where the elements of function $\mathbf{f}$ give the unit prices or inverse demand functions and C_k is the cost function of firm k. It is assumed that the cost of each firm depends on its own production levels, and the unit prices depend on the total production levels of all products.

The Nash (or Cournot) equilibrium point of the N-person noncooperative game with sets S_k of strategies and payoff functions φ_k is any vector $(\mathbf{x}_1^*, \ldots$ $\ldots, \mathbf{x}_N^*)$ such that for all k, $\mathbf{x}_k^* \in S_k$ and for arbitrary k and $x_k \in S_k$,

$$\varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_{k-1}^*, \mathbf{x}_k, \mathbf{x}_{k+1}^*, \ldots, \mathbf{x}_N^*) \leq \varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_{k-1}^*, \mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \ldots, \mathbf{x}_N^*). \tag{4}$$

The classical Cournot oligopoly without product differentiation is a special case of the above game obtained by letting $M = 1$. The existence, uniqueness and stability of the Cournot equilibrium for this special model has been extensively investigated. Okuguchi [5] presents a detailed background for the contributions. Szidarovszky [8], Szidarovszky and Yakowitz [9] present a new proof for the existence and uniqueness of the Cournot oligopoly equilibrium without product differentiation. If $M = N$, and if, in addition, each firm produces non-identical product, then the Cournot oligopoly under product differentiation and with single product firms is obtained. This model has also been much discussed (see Okuguchi [5]). The Cournot oligopoly with multi-product firms where all firms produce in general more than one product has so far not been systematically analyzed, some results by Eichhorn [2, 3], Selten [7], Bird [1] and Szidarovszky [10] are known for simple cases where demand and cost functions are assumed to be linear. Moreover, most of these authors have been concerned with price-adjusting firms, instead of as assumed here, quantity-adjusting firms.

This paper proceeds as follows. In Section 2 we will establish relations between equilibrium points of the N-person noncooperative games and the fixed points of a special, low dimensional point-to-set mapping. In Section 3 we will present and prove an existence theorem of the Cournot equilibrium point for the oligopoly with multi-product firms. The condition for the theorem will then be interpreted especially in relation to the properties of the demand functions. In Section 4 conclusions are summarized.

2. Equilibrium points and fixed points of point-to-set mappings

There is a remarkable relation between equilibrium points of N-person noncooperative games and fixed points of certain point-to-set mappings. Introduce the following notation:

$$\Phi(\mathbf{x}, \mathbf{y}) \equiv \sum_{k=1}^{N} \varphi_k(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_N), \tag{5}$$

where for all k, $\mathbf{x}_k$ and $\mathbf{y}_k \in S_k$. Define the point-to-set mapping as follows:

$$H(\mathbf{x}) = \left\{\mathbf{u} \mid \mathbf{u} \in \prod_{k=1}^{N} S_k, \Phi(\mathbf{x}, \mathbf{u}) = \max_{y \in \prod_{k=1}^{N} S_k} \Phi(\mathbf{x}, \mathbf{y})\right\} \tag{6}$$

Then the following result is true (see e.g. Szidarovszky [10]):

Theorem. 1 *A vector* $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ *is an equilibrium point of the N-person noncooperative game if and only if* $\mathbf{x}^* \in H(\mathbf{x}^*)$, *that is,* $\mathbf{x}^*$ *is a fixed point of mapping* $H(\mathbf{x}^*)$.

If one wishes to apply this theorem to the Cournot oligopoly with multiproduct firms, the fixed points of a point-to-set mapping of the dimension NM have to be determined. In the remainder of this section a similar result will be proven when the dimension of the mapping equals M. That is, the dimension can be drastically reduced.

Let

$$\varphi_k(\mathbf{s}, \mathbf{x}_k, \mathbf{t}_k) \equiv \mathbf{t}_k' \cdot \mathbf{f}(\mathbf{s} - \mathbf{x}_k + \mathbf{t}_k) - C_k(\mathbf{t}_k) \tag{7}$$

denote the k-th firm's profit function, if the total production levels are given as vector $\mathbf{s}$ and if, subsequently this firm changes its own output vector from $\mathbf{x}_k$ to $\mathbf{t}_k$. Define next the M-dimensional point-to-set mapping

$$H_k(\mathbf{s}) = \{\mathbf{u}_k \mid \mathbf{u}_k \in S_k,\ \varphi_k(\mathbf{s}, \mathbf{x}_k, \mathbf{u}_k) = \max_{\mathbf{t}_k \in S_k} \varphi_k(\mathbf{s}, \mathbf{u}_k, \mathbf{t}_k)\} \tag{8}$$

for each firm k and given total production vector $\mathbf{s}$. Let, in addition,

$$H(\mathbf{s}) \equiv \left\{\mathbf{t} \mid \text{there exists } \mathbf{x}_k \in H_k(\mathbf{s})\ (\forall\, k), \text{ such that } \mathbf{t} = \sum_{k=1}^{N} \mathbf{x}_k\right\}. \tag{9}$$

From the construction of the point-to-set mappings H_k and H one may easily prove the following result:

Theorem 2. *A strategy vector* $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ *is a Cournot equilibrium point for the oligopoly game with multiproduct firms if and only if* $\mathbf{s}^* = \sum_{k=1}^{N} \mathbf{x}_k^*$ *is a fixed point of mapping H, and for all k,* $\mathbf{x}_k^* \in H_k(\mathbf{s}^*)$.

In comparing the fixed point problems given by Theorems 1 and 2, the following remarks can be made:

1. The dimension of mapping (9) equals M, which is usually much smaller than that of mapping (6).

2. Any vector $\mathbf{u}$ belongs to $H(\mathbf{x})$, defined by (6), if and only if the first order conditions for the optimality of $\varphi(\mathbf{x}, \mathbf{u})$ hold with fixed $\mathbf{x}$. In the case of mapping (9), the first order conditions for the optimality of each functions $\varphi_k(\mathbf{s}, \mathbf{x}_k, \mathbf{t}_k)$ should be verified, and then the condition $\mathbf{s} = \sum_{k=1}^{N} \mathbf{x}_k$ must be checked. In comparing the two processes, in both cases all vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$ must be considered: in the case of (6) they are simultaneously investigated (that is, a large set of equations and inequalities must be solved), and in the case of mapping (9) vectors $\mathbf{x}_k$ are examined almost independently, since they are connected only via their sum $\mathbf{s}$ (that is a much smaller set of equations and inequalities must be solved). For the numerical solution of the resulted system of equations, the methods discussed in [12] can be applied.

3. An existence theorem

In this section an existence theorem for the Cournot equilibrium point for the oligopoly game with multi-product firms will be introduced. Let us introduce the following assumptions on the demand and cost functions:

A) There exists a convex, closed set $D \subset R^M_+$ such that for $\mathbf{s} \notin D$, necessarily $\mathbf{f}(\mathbf{s}) = \mathbf{0}$.

B) Each element of $\mathbf{f}$ is continuously differentiable, concave and for arbitrary $\mathbf{s} \in S \equiv \prod_{k=1}^{N} S_k$, matrix $\mathcal{J}_\mathbf{f}(\mathbf{s}) + \mathcal{J}'_\mathbf{f}(\mathbf{s})$ is negative semidefinite, where $\mathcal{J}_\mathbf{f}$ denotes the Jacobian matrix of function $\mathbf{f}$.

C) For arbitrary $\mathbf{s} \in D$ and $\mathbf{0} \leq \tilde{\mathbf{s}} \leq \mathbf{s}$, necessarily $\tilde{\mathbf{s}} \in D$.

D) Cost functions C_k $(1 \leq k \leq N)$ are continuous, convex and strictly increasing in each variable.

E) The set S of simultaneous strategies is convex, closed and for all $\mathbf{x} \in S$ and $\mathbf{0} \leq \tilde{\mathbf{x}} \leq \mathbf{x}$ necessarily $\mathbf{x} \in S$.

Before presenting and proving our existence theorem, let us prove the following result.

Lemma 1. *Consider the vector function* $\mathbf{g}$, *where* $\mathcal{R}(\mathbf{g}) \subset R^M_+$, *and* $\mathcal{D}(\mathbf{g})$ *is a convex set in* R^M_+. *Assume that each element of* $\mathbf{g}$ *is concave and continuously differentiable. Let* $\mathcal{J}_g$ *denote the Jacobian of* $\mathbf{g}$ *and assume that for arbitrary* $\mathbf{x} \in \mathcal{D}(\mathbf{g})$, *matrix* $\mathcal{J}_\mathbf{g}(\mathbf{x}) + \mathcal{J}'_\mathbf{g}(\mathbf{x})$ *is negative semidefinite. Then the function*

(10) $$h(\mathbf{x}) = \mathbf{x}'\mathbf{g}(\mathbf{x})$$

is concave.

Proof. Let ∇ denote the gradient operator, then

(11) $$\nabla h(\mathbf{x}) = \mathbf{g}(\mathbf{x})' + \mathbf{x}'\mathcal{J}_\mathbf{g}(\mathbf{x}).$$

Since the elements of $\mathbf{g}$ are concave, for arbitrary $\mathbf{x}, \mathbf{y} \in \mathcal{D}(\mathbf{g})$, we have

(12) $$\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{x}) \leq \mathcal{J}_\mathbf{g}(\mathbf{x})\ (\mathbf{y} - \mathbf{x}).$$

The negative semidefinite property of matrix $\mathcal{J}_\mathbf{g}(\mathbf{x}) + \mathcal{J}'_\mathbf{g}(\mathbf{x})$ implies that

(13) $$0 \geq \frac{1}{2}(\mathbf{y}-\mathbf{x})'[\mathcal{J}_\mathbf{g}(\mathbf{x}) + \mathcal{J}'_\mathbf{g}(\mathbf{x})](\mathbf{y}-\mathbf{x}) = (\mathbf{y}-\mathbf{x})'\mathcal{J}'_\mathbf{g}(\mathbf{x})(\mathbf{y}-\mathbf{x}),$$

from which we can conclude that

(14) $$\mathbf{y}'[\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{x})] \leq \mathbf{y}'\mathcal{J}_\mathbf{g}(\mathbf{x})(\mathbf{y}-\mathbf{x}) = (\mathbf{y}-\mathbf{x})'\mathcal{J}'_\mathbf{g}(\mathbf{x})\mathbf{y} \leq (\mathbf{y}-\mathbf{x})'\mathcal{J}_\mathbf{g}(\mathbf{x})\mathbf{x}.$$

And finally, by simple calculation,

(15) $$(\mathbf{y}-\mathbf{x})'[\mathbf{g}(\mathbf{x}) + \mathcal{J}'_\mathbf{g}(\mathbf{x})\mathbf{x}] \geq \mathbf{y}'\mathbf{g}(\mathbf{y}) - \mathbf{x}'\mathbf{g}(\mathbf{x}),$$

that is,

(16) $$h(\mathbf{y})-h(\mathbf{x})\leq \nabla h(\mathbf{x})(\mathbf{y}-\mathbf{x})$$

which proves the lemma. □

Our existence theorem is stated as follows:

Theorem 3. *Under assumptions A)–E), the Cournot oligopoly game with multi-product firms has at least one equilibrium point.*

Proof. We shall proceed in three steps.

a) First we shall verify that if $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ is an equilibrium point, then

(17) $$\mathbf{s}^* = \sum_{k=1}^{N} \mathbf{x}_k^* \in D.$$

In contrary to the assertion assume that $\mathbf{s}^* \notin D$. If D is empty, then $\mathbf{f}(\mathbf{s})\equiv\mathbf{0}$, consequently $\mathbf{x}^* = \mathbf{0}$ is the only equilibrium point. Otherwise, at least one element of $\mathbf{s}^*$ is positive and there exist indices p and q such that the q-th element $x_p^{(q)*}$ of vector $\mathbf{x}_p^*$ is positive. Since set D is closed, there exists an $x_p^{(q)}$ such that $0\leq x_p^{(q)}\leq x_p^{(q)*}$ and $\mathbf{s}^*+\mathbf{e}_q(x_p^{(q)}-x_p^{(q)*})\notin D$.
Then vector $\mathbf{x} = (\mathbf{x}_1^*, \ldots, \mathbf{x}_{p-1}^*, \mathbf{x}_p, \mathbf{x}_{p+1}^*, \ldots, \mathbf{x}_N^*)$ for which $\mathbf{x}_p = \mathbf{x}_p^* + \mathbf{e}_q(x_p^{(q)}-x_p^{(q)*})$ satisfies

(18) $$\varphi_p(\mathbf{x}) = -C_p(\mathbf{x}_p) > -C_p(\mathbf{x}^*) = \varphi_p(\mathbf{x}^*),$$

contradicting to the definition of $\mathbf{x}^*$ as an equilibrium. Note that $\mathbf{e}_q$ denotes the q-th unit vector.

b) Consider the N-person game with original payoff functions and sets of strategies but with restricted set of dimultaneous strategies

(19) $$\hat{S} = \left\{\mathbf{x}|\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \text{ such that } \mathbf{x}_k\in S_k(\forall k),\ \sum_{k=1}^{N} \mathbf{x}_k\in D\right\}.$$

Then the previous step of the proof implies that if $\mathbf{x}^*$ is an equilibrium point of the original game, then it is also an equilibrium point of the restricted game. Let $\mathbf{x}^*=(\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ be an equilibrium point of the restricted game. We shall prove that $\mathbf{x}^*$ is an equilibrium point of the original game, too. Let k and $\mathbf{x}_k\in S_k$ arbitrary. If $\mathbf{x} = (\mathbf{x}_1^*, \ldots, \mathbf{x}_{k-1}^*, \mathbf{x}_k, \mathbf{x}_{k+1}^*, \ldots, \mathbf{x}_N^*)\notin\hat{S}$, then

(20) $$\varphi_k(\mathbf{x}) = -C_k(\mathbf{x}_k)\leq -C_k(\mathbf{0}) =$$
$$= \varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_{k-1}^*\, \mathbf{0}, \mathbf{x}_{k+1}^*, \ldots, \mathbf{x}_N^*)\leq\varphi_k(\mathbf{x}^*),$$

since

$$(\mathbf{x}_1^*, \ldots, \mathbf{x}_{k-1}^*, \mathbf{0}, \mathbf{x}_{k+1}^*, \ldots, \mathbf{x}_N^*)\in\hat{S}.$$

c) By applying Lemma 1, we can conclude that the reduced game has at least one equilibrium point, since it satisfies all of the conditions for the Nikaido-Isoda [4] theorem on convex games. This equilibrium point coincides with the Cournot equilibrium point of the original game. □

Some remarks on Theorem 3 might be in order. This theorem is a generalization of the existence part of Theorem 1 in Szidarovszky [8] and Szidarovszky and Yakowitz [11], since in the one dimensional case $\mathcal{J}_f(\mathbf{x})$ is the derivative of $f(\mathbf{x})$. Assumptions *B)* and *D)* together imply that each firm's profit function is concave in its own production level. The existence of equilibrium point can be seen to be ensured if we assume, alternatively and more generally, that each firm's profit function is concave in its own production level.

In order to clarify the economic implications of assumption *B)*, let

$$\mathcal{J}_f(\mathbf{s}) \equiv [f_i^j(s)],$$

where f_i is the i-th element of f, and f_i^j is the partial derivative of f_i with respect to change in the j-th element of $\mathbf{s}$. We can reasonably assume that

$$f_i^i(\mathbf{s}) < 0, \; f_i^j(\mathbf{s}) \leq 0, \; i \neq j \tag{21}$$

for $\forall$ i, j and for arbitrary $\mathbf{s} \in S$.
Assume furthermore that

$$|2f_i^i(\mathbf{s})| > \sum_{j \neq i} |f_i^j(\mathbf{s}) + f_j^i(\mathbf{s})| \tag{22}$$

for arbitrary $\mathbf{s} \in S$ and $\forall$ i.
Under (22), $\mathcal{J}_f(\mathbf{s}) + \mathcal{J}_f^T(\mathbf{s})$ becomes a negative dominant diagonal matrix, hence it is negative definite. Thus, (22) is satisfied if e.g.

$$|f_i^i(\mathbf{s})| > \sum_{j \neq i} |f_i^j(\mathbf{s})| \tag{23a}$$

$$|f_i^i(\mathbf{s})| > \sum_{j \neq i} |f_j^i(\mathbf{s})| \tag{23b}$$

are simultaneously satisfied for arbitrary $\mathbf{s} \in S$ and $\forall$ i.

Inequality (23a) implies that for arbitrary i the effect of change in the total output of the i-th product on its price dominate in a sense the change in the price of the same product caused by simultaneous and identical changes in the outputs of all other products. Inequality (23b) shows that for any i, the effect of change in the total output of the i-th product on its price dominates the sum of changes it induces to prices of all other products.

4. Conclusion

We have formulated a Cournot oligopoly game with multiproduct firms and presented an existence theorem for the equilibrium point after showing relations between the Nash equilibrium point for the N-person noncooperative game and the fixed points of point-to-set mappings. The condition for the existence theorem has been interpreted in relation to the properties of the demand functions. If $M = 1$, our model reduces to a classical Cournot oligopoly model without product differentiation. If $M = N$, and if, in addition, each firm produces non-identical product, our model becomes one of oligopoly under product differentiation and with single product firms. Finally, note that a quantity-adjusting oligopoly problem has been investigated and not a price-adjusting one.

REFERENCES

[1] *Bird C. G.:* On the stability of a class of noncooperative games. *Linear Algebra and its Applications* **40** (1981), 119 – 127.
[2] *Eichhorn W.:* Zur Statischen Theorie des Mehrproduktoligopols. *Operations Research Verfahren* **10** (1971), 16 – 33.
[3] *Eichhorn W.:* Zur Dynamischen Theorie des Mehrproduktoligopols. *Jahrbücher für Nationalökonomie und Statistik* 1972, 478 – 515.
[4] *Nikaido H.* and *Isoda K.:* Note on noncooperative convex games. *Pacific Journal of Mathematics,* **5** (1955), 807 – 815.
[5] *Okuguchi K.:* Expectations and Stability in Oligopoly Models. Springer Verlag, Berlin/ Heidelberg/New York, 1976.
[6] *Rosen J, B.:* Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica* **33** (1965), 520 – 534.
[7] *Selten, R.: Preispolitik der Mehrproduktunternehmung in der statischen Theorie.* Springer Verlag, Berlin/Heidelberg/New York, 1970.
[8] *Szidarovszky, F.:* On the oligopoly game, *Department of Mathematics, K. Marx University,* 1970 – 1, Budapest.
[9] *Szidarovszky, F.* and *S. Yakowitz:* A new proof of the existence and uniqueness of the Cournot equilibrium. *International Economic Review* **18** (1977), 787 – 789.
[10] *Szidarovszky, F.: Game Theory.* Tankönyvkiadó Budapest, 1978 (in Hungarian).
[11] *Szidarovszky, F.* and *S. Yakowitz:* Contributions to cournot oligopoly theory. *Journal of Economic Theory* **28** (1982), 51 – 70.
[12] *Szidarovszky, F.* and *S. Yakowitz: Principles and Procedures of Numerical Analysis.* Plenum Publ. Comp., New York/ London, 1978.