(0, 1, 4) LACUNARY INTERPOLATION BY SPLINES

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1. Introduction. Recently, Th. Fawzy ([1]–[4]) presented some local methods to solve (0,3), (0, 2, 3) and (0, 2) lacunary interpolation problems using g-splines. A. Meir and A. Sharma [6], B. K. Swartz and R. S. Varga [9], A. K. Varma [10], R. S. Mishra and K. K. Mather [7] presented global methods for solving such lacunary interpolation problems.

In this paper, we follow the methods introduced by Th. Fawzy and present a spline solution for the (0, 1, 4) lacunary interpolation problem. We are given the mesh points:

$$\triangle: 0 = x_0 < x_1 < \ldots < x_n = 1$$

with $x_{k+1} - x_k = h$, $k = 0, 1, \ldots, n-1$, and real numbers

$$\{f_k, f'_k, f_k^{(4)}\}_{k=0}^n.$$

associated with the knots.

We are going to construct spline interpolant $S_\triangle(x)$ for which $S_\triangle^{(q)}(x_i) = f_i^{(q)}$ $i = 0, 1, \ldots, n$ and $q = 0, 1, 4$. This construction is given in the following two cases.

2. Case A. In this case we consider the situation when $Y \in C^4[0, 1]$ and then we define the spline interpolant as follows:

$$S_\triangle(x) = S_k(x) = y_k + y'_k(x-x_k) + \frac{1}{2} a_k(x-x_k)^2 + \frac{1}{3!} b_k(x-x_k)^3 + \frac{1}{4!} y_k^{(4)}(x-x_k)^4, \tag{2.1}$$

where $x_k \le x \le x_{k+1}$ and $k = 0, 1, \ldots, n-1$. If we require that $S_\triangle(x) \in C^1[0, 1]$, then it is easy to prove that formula (2.1) assures that this spline polynomial exists and is unique. That is clear from the continuity conditions of $S_\Delta(x)$ and $S'_\Delta(x)$ from which we get:

$$y_{k+1} = y_k + y'_k h + \frac{1}{2} a_k h^2 + \frac{1}{3!} b_k h^3 + \frac{1}{4!} y_k^{(4)} h^4. \tag{2.2}$$

and

$$y'_{k+1} = y' + a_k h + \frac{1}{2} b_k h^2 + \frac{1}{3!} y_k^{(4)} h^3. \tag{2.3}$$

Hence,

$$a_k = \frac{6}{h^2} A_k - \frac{2}{h} B_k, \tag{2.4}$$

$$b_k = \frac{6}{h^2} B_k - \frac{12}{h^3} A_k, \tag{2.5}$$

$$A_k = y_{k+1} - y_k - h y'_k - \frac{h^4}{4!} y_k^{(4)} \text{ and} \tag{2.6}$$

$$B_k = y'_{k+1} - y'_k - \frac{h^3}{3!} y_k^{(4)}. \tag{2.7}$$

Theorem 1. *Let $S_k(x)$ be the spline interpolant solving the lacunary case (0, 1, 4) for which $y(x) \in C^4[0, 1]$. Then for all $x \in [0, 1]$ the inequality*

$$|y^{(i)}(x) - S_\Delta^{(i)}(x)| \leq c_i h^{4-i} \omega(h)$$

holds for all $i = 0, 1, 2, 3$ and 4, where $\omega(h)$ is the modulus of continuity of $y^{(4)}(x)$, $c_0 = 7/12$, $c_1 = 3/2$, $c_2 = 31/12$, $c_3 = 5/2$ and $c_3 = 1$.

Proof. From (2.4) we find

$$|a_k - y''_k| = \left| \frac{6}{h^2}\left(y_{k+1} - y_k - h y'_k - \frac{h^4}{4!} y_k^{(4)}\right) - \frac{2}{h}\left(y'_{k+1} - y'_k - \frac{h^2}{3!} y_k^{(4)}\right) - y''_k \right|,$$

and taking:

$$y_{k+1} = y_k + h y'_k + \frac{h^2}{2!} \bar{y}''_k + \frac{h^3}{3!} y'''_k + \frac{h^4}{4!} y^{(4)}(\eta_k),$$

where $x_k < \eta < x_{k+1}$ and

$$y'_{k+1} = y'_k + y''_k \cdot h + \frac{h^2}{2} y'''_k + \frac{h^3}{3!} y^{(4)}(\xi_k),$$

where $x_k < \xi_k < x_{k+1}$, it is easy to get:

$$|a_k - y''_k| \leq \frac{7}{12} h^2 \omega(h). \tag{2.8}$$

Similarly, from (2.7) we get

$$(2.9)\qquad |b_k - y_k'''| = \left|\frac{6}{h^2}\left(y_{k+1}' - y' - \frac{h^3}{3!}y_k^{(4)}\right) - \frac{12}{h^3}\left(y_{k+1} - y_k - hy_k' - \frac{h}{4!}y_k^{(4)}\right) - y_k'''\right| \le \frac{3}{2}h\omega(h),$$

and using (2.1), (2.8) and (2.9) we get, for $x_k \le x \le x_{k+1}$ and $k = 0, 1, \ldots, n-1$:

$$|S_k(x) - y(x)| = \Big| y_k + y_k'(x-x_k) + \frac{1}{2}a_k(x-x_k)^2 + \frac{1}{3!}b_k(x-x_k)^3 + \frac{1}{4!}y_k^{(4)}(x-x_k)^4 - y_k - y_k'(x-x_k) - \frac{1}{2}y_k''(x-x_k)^2 - \frac{1}{3!}y_k'''(x-x_k)^3 - \frac{1}{4!}y^{(4)}(\eta_k)(x-x_k)^4 \Big| \le$$

$$\le \frac{h^2}{2}|a_k - y_k''| + \frac{h^3}{3!}|b_k - y_k'''| + \frac{h^4}{3!}h^3\omega(h) \le \frac{7}{12}h^4\omega(h).$$

Similarly:

$$|S_k'(x) - y'(x)| \le h|a_k - y_k''| + \frac{h^2}{2}|b_k - y_k'''| + \frac{h^3}{3!}h^3\omega(h) \le \frac{3}{2}h^3\omega(h),$$

$$|S_k''(x) - y''(x)| \le |a_k - y_k''| + h|b_k - y_k'''| + \frac{h^2}{2}\omega(h) \le \frac{31}{12}h^2\omega(h),$$

$$|S_k'''(x) - y_k'''(x)| \le |b_k - y_k'''| + h\omega(h) \le \frac{5}{2}h\omega(h), \quad \Box$$

and finally

$$|S_k^{(4)}(x) - y^{(4)}(x)| = |y_k^{(4)} - y^{(4)}(x)| \le \omega(h). \quad \Box$$

3. Case B. In this case $y \in C^5[0, 1]$ and we define the spline interpolant as

$$(3.1)\qquad S_\Delta(x) = S_k(x) = y_k + y_k'(x-x_k) + \frac{1}{2}a_k(x-x_k)^2 + \frac{1}{3!}b_k(x-x_k)^3 + \frac{1}{4!}y_k^{(4)}(x-y_k)^4 + \frac{1}{5!}c_k(x-x_k)^5,$$

where $x \in [x_k, x_{k+1}]$ and $k = 0, 1, \ldots, n-1$.
Let

$$(3.2)\qquad c_k = \frac{1}{h}(y_{k+1}^{(4)} - y_k^{(4)}).$$

Now if $S_\Delta(x)\in C^1[0, 1]$, then the existence and uniqueness of $S_\Delta(x)$ is easy to be proved, since in this case a_k and b_k are uniquely determined by

$$a_k = \frac{6}{h^2}A_k - \frac{2}{h}B_k, \tag{3.3}$$

$$b_k = \frac{12}{h^3}\left(\frac{h}{2}B_k - A_k\right), \tag{3.4}$$

$$A_k = y_{k+1} - y_k - hy_k' \frac{h^4}{4!}y_k^{(4)} - \frac{h^5}{5!}b_k \text{ and} \tag{3.5}$$

$$B_k = y_{k+1}' - y_k' - \frac{h^3}{3!}y_k^{(4)} - \frac{h^4}{4!}c_k. \tag{3.6}$$

Theorem 2. *Let $S_\Delta(x)$ be the spline interpolant, given in* (3.1), *and solving the lacunary case* (0, 1, 4) *for which $y\in C^5[0, 1]$. Then for all $x\in[0, 1]$ the inequality:*

$$|S_\Delta^{(4)} - y^{(i)}(x)| \le c_i h^{5-i}\omega(h)$$

holds true for all $i = 0, 1, \ldots, 5$, where $\omega(h)$ is the mudulus of continuity of $y^{(5)}(x)$ and

$$c_0 = 2/15,\ c_1 = 19/60,\ c_2 = 13/20,\ c_3 = 17/20,\ c_4 = 1,\ c_5 = 1.$$

Proof. From (3.3), (3.5) and (3.6) we get

$$|a_k - y_k''| \le \frac{2}{15}h^3\omega(h) \tag{3.7}$$

and from (3.4), (3.5) and (3.6) we can get

$$|b_k - y_k'''| \le 7/20h^2\omega(h). \tag{3.8}$$

Thus:

$$|S_k(x) - y(x)| = |y_k - y_k'(x-x_k) + \frac{1}{2}a_k(x-x_k)^2 + \frac{1}{3!}b_k(x-x_k)^3 -$$

$$+\frac{1}{4!}y_k^{(4)}(x-x_k)^4 + \frac{1}{5!}c_k(x-x_k)^5 - y_k - y_k'(x-x_k) -$$

$$-\frac{1}{2}y_k''(x-x_k)^2 - \frac{1}{3}y_k'''(x-x_k)^3 - \frac{1}{4!}y_k^{(4)}(x-x_k) -$$

$$-\frac{1}{5!}y^{(5)}(\xi_k)(x-x_k)^5| \le \frac{h^2}{2}|a_k - y_k''| + \frac{h^3}{3!}|b_k - y_k'''| +$$

$$+\frac{2}{15}h^5\omega(h).$$

Similarly, it is easy to complete the proof for $|S_k^{(i)}(x) - y^{(i)}(x)|$ where $i = 1, 2, \ldots, 5$. □

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