INTERVAL FILLING SEQUENCES

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(Received July 16, 1984)

1. The notion of interval filling sequence has been introduced in the paper [4]. Let Λ denote the set of those real sequences $\{\lambda_n\}$, for which

$$\lambda_n > \lambda_{n+1} > 0 \ (n \in \mathbf{N}) \text{ and } L := \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Definition 1.1. We call the sequence $\{\lambda_n\} \in \Lambda$ *interval filling,* if for any $x \in [0, L]$ there exists a sequence $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$ so that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \tag{1.1}$$

In [4] we have proved the following

Theorem 1.1. *The sequence* $\{\lambda_n\} \in \Lambda$ *is interval filling if and only if*

$$\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i$$

for any $n \in \mathbf{N}$.

In this paper we investigate certain classes of interval filling sequences, in order to clarify the finer structure of the representations (1.1).

2. We shall need the following definitions:

Definition 2.1. Let $\{\lambda_n\} \in \Lambda$ be a fixed sequence and $k \in \mathbf{N}$. We shall call the number $x \in [0, L]$ *k-decomposable* if for any $x_i \in [0, L]$ $(i = 1, 2, \ldots, k)$ satisfying $x = \sum_{i=1}^{k} x_i$ there exist sequences $\varepsilon_n(i) \in \{0, 1\}$ $(n \in \mathbf{N}, i = 1, 2, \ldots, k)$ so that

$$x_i = \sum_{n=1}^{\infty} \varepsilon_n(i) \lambda_n \quad (i = 1, 2, \ldots, k) \tag{2.1}$$

and $\sum_{i=1}^{k} \varepsilon_n(i) \in \{0,1\}$ $\quad (n \in \mathbf{N})$.

Definition 2.2. Let $\{\lambda_n\}\in\Lambda$ and $k\in\mathbf{N}$ fixed. We call the sequence *interval filling of order k* if any $x\in[0, L]$ is k-decomposable.

Remarks. (i) If the sequence $\{\lambda_n\}\in\Lambda$ is interval filling of order $k\geqq 2$, then it is also interval filling of order $(k-1)$. As a matter of fact, if any $x\in[0, L]$ is k-decomposable, then choosing the numbers $x_i\in[0, L]$ $(i = 1, 2, \dots$ $\dots, k)$ so as to have $x = \sum_{i=1}^{k-1} x_i$ and $x_k := 0$ we obtain that x is $(k-1)$-decomposable.

(ii) Clearly, the sequence $\{\lambda_n\}\in\Lambda$ is interval filling of the first order iff it is interval filling in the sense of Definition 1.1.

Theorem 2.1. *The sequence $\{\lambda_n\}\in\Lambda$ is interval filling of order k, if and only if*

$$k\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i \tag{2.2}$$

holds for any $n\in\mathbf{N}$.

Proof. (i) Let $\{\lambda_n\}\in\Lambda$ be interval filling of order k (for a fixed $k\in\mathbf{N}$), and suppose that in contradiction to our statement (2.2) does not hold for some $n\in\mathbf{N}$, i.e. that

$$k\lambda_n > \sum_{j=n+1}^{\infty} \lambda_j.$$

Then there exist numbers $x_1, x_2, \dots, x_{k-1}$ and x_k such that

$$\frac{1}{k}\sum_{j=n+1}^{\infty} \lambda_j < x_i < \min\left\{\lambda_n, \frac{1}{k}\sum_{j=n}^{\infty}\lambda_j\right\} \tag{2.3}$$

for $i = 1, 2, \dots, k-1$ and

$$\sum_{j=1}^{n-1}\lambda_j + \frac{1}{k}\sum_{j=n+1}^{\infty}\lambda_j < x_k < \min\left\{\sum_{j=1}^{n}\lambda_j, \sum_{j=1}^{n-1}\lambda_j + \frac{1}{k}\sum_{j=n}^{\infty}\lambda_j\right\}. \tag{2.4}$$

Now $x_1+x_2+\dots+x_k<L$ and by (2.3) the values $\lambda_1, \lambda_2, \dots, \lambda_n$ cannot occur in the representation (2.1) of the numbers $x_1, x_2, \dots, x_{k-1}$ (i.e. $\varepsilon_j(i) = 0$ for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, k-1$). On the other hand, in view of (2.4) the representation (2.1) of x_k cannot contain all the values $\lambda_1, \lambda_2, \dots, \lambda_n$ so that

$$x_1+x_2+\dots+x_k \leqq \sum_{\substack{j=1\\ j\neq n}}^{\infty}\lambda_j$$

follows. If we still add the left-hand sides of the inequalities (2.3) and (2.4), we get

$$x_1+x_2+\dots+x_k > \sum_{\substack{j=1\\ j\neq n}}^{\infty}\lambda_j$$

a contradiction. Thus the number $x := \sum_{i=1}^{k} x_i$ is not k-decomposable.

(ii) Let us suppose that the sequence $\{\lambda_n\}\in\Lambda$ satisfies the inequalities (2.2). Let moreover $x_i\in[0, L]$ $(i = 1, 2, \ldots, k)$ be arbitrary numbers for which $\sum_{i=1}^{k} x_i \leqq L$.

We now define the numbers $\varepsilon_n(i)$ $(n\in\mathbf{N}, i = 1, 2, \ldots, k)$ inductively as follows:

$$\varepsilon_n(i) = \begin{cases} 1 \text{ if } \varepsilon_n(j) = 0 \text{ for } j<i \text{ and} \\ \quad \sum_{l=1}^{n-1} \varepsilon_l(i)\lambda_l + \lambda_n \leqq x_i, \\ 0 \text{ otherwise.} \end{cases}$$

It is clear from the definition that $\sum_{n=1}^{k} \varepsilon_n(i)\in\{0,1\}$ $(n\in\mathbf{N})$ and

$$\sum_{n=1}^{\infty} \varepsilon_n(i)\lambda_n \leqq x_i \quad (i = 1, 2, \ldots, k). \tag{2.5}$$

We are going to show that in (2.5) equality holds. We distinguish three cases:

$$\text{(a)} \qquad \sum_{i=1}^{k} \varepsilon_n(i) = 1 \quad \text{for every} \quad n\in\mathbf{N}.$$

If, for some $i\in\{1, 2, \ldots, k\}$, strict inequality would hold in (2.5), then condition (a) would imply

$$L = \sum_{n=1}^{\infty} \lambda_n < x_1 + x_2 + \ldots + x_k$$

in contradiction to the choice of the numbers x_i.

(b) There exist finitely many values $n\in\mathbf{N}$ for which

$$\sum_{i=1}^{k} \varepsilon_n(i) = 0.$$

Let us show that this is impossible. Indeed, suppose that (b) is valid, and let N be the greatest natural number for which $\sum_{i=1}^{k} \varepsilon_N(i) = 0$. Then

$$\sum_{l=1}^{N-1} \varepsilon_l(i)\lambda_l + \lambda_N > x_i$$

for $i = 1, 2, \ldots, k$ and from this

$$\sum_{i=1}^{k} \sum_{l=1}^{\infty} \varepsilon_l(i)\lambda_l = \sum_{i=1}^{k} \sum_{l=1}^{N-1} \varepsilon_l(i)\lambda_l + \sum_{l=N+1}^{\infty} \lambda_l \geqq$$

$$\geqq \sum_{i=1}^{k} \left(\sum_{l=1}^{N-1} \varepsilon_l(i)\lambda_l + \lambda_N \right) > x_1 + x_2 + \ldots + x_k$$

follows, a contradiction.

(c) There exist infinitely many values $n \in \mathbf{N}$ for which

$$\sum_{i=1}^{k} \varepsilon_n(i) = 0.$$

For these values of n

$$x_i < \sum_{l=1}^{n-1} \varepsilon_l(i)\lambda_l + \lambda_n$$

and if $n \to \infty$ then by $\lambda_n \to 0$ this implies

$$x_i \leqq \sum_{l=1}^{\infty} \varepsilon_l(i)\lambda_l.$$

i.e. in (2.5) equality holds. □

Remark. For $k = 1$ Theorem 2.1 is clearly equivalent with Theorem 1.1.

Examples. (1) Let $1 < q \leqq 2$. Then the sequence $\lambda_n := \frac{1}{q^n}$ belongs to Λ.

Proposition. *The sequence* $\left\{ \frac{1}{q^n} \right\} \in \Lambda$ *is interval filling of order k if and only if*

$$1 < q \leqq \frac{k+1}{k}.$$

Proof. In view of Theorem 2.1 we must investigate, when will the inequality

$$\frac{k}{q^n} \leqq \sum_{j=n+1}^{\infty} \frac{1}{q^j} = \frac{1}{q^n} \frac{1}{q-1}$$

be satisfied for any $n \in \mathbf{N}$: (2.6) immediately follows. □

(2) Let $N \in \mathbf{N}$ be fixed, and consider the sequence

$$\lambda_n^{(N)} := \frac{1}{(N+n)^2} \quad (n \in \mathbf{N}).$$

Proposition. *For any $k \in \mathbf{N}$ there exists $N(k) \in \mathbf{N}$ so that the sequence $\{\lambda_n^{(N(k))}\} \in \Lambda$ is interval filling of order k.*

Proof. In view of Theorem 2.1 it suffices to show that for $k \in \mathbf{N}$ fixed

$$\frac{k}{(N+n)^2} \leqq \sum_{i=n+1}^{\infty} \frac{1}{(N+i)^2}$$

holds for any $n \in \mathbf{N}$, if only N is large enough. An investigation of the function $x \to \frac{x^2}{x+1}$ shows that for any $k \in \mathbf{N}$ there exists such an x_0 that $x > x_0$ implies $\frac{x^2}{x+1} > k$. Choosing $N = N(k) > x_0$ we now infer that

$$\frac{(N+n)^2}{N+n+1} > k$$

i.e.

$$\frac{k}{(N+n)^2} < \frac{1}{N+n+1}$$

holds for any $n \in \mathbf{N}$. From this

$$\frac{k}{(N+n)^2} < \frac{1}{N+n+1} - \frac{1}{N+n+2} + \frac{1}{N+n+2} - \frac{1}{N+n+3} + \dots =$$

$$= \sum_{i=n+1}^{\infty} \frac{1}{(N+i)(N+i+1)} \leqq \sum_{i=n+1}^{\infty} \frac{1}{(N+i)^2}$$

i.e. $\{\lambda_n^{(N)}\}$ is interval filling of order k. □

3. Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence.

Definition 3.1. The function $F: [0, L] \to \mathbf{R}$ is said to be *completely additive* if for any sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbf{N}$) the equality

$$F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n) \tag{3.1}$$

holds.

We have investigated completely additive functions in the papers [3] and [4]. A more general form of the following theorem can be found in [4], but the method of proof is quite different.

Theorem 3.1. *If $\{\lambda_n\} \in \Lambda$ is an interval filling sequence of second order and $F: [0, L] \to \mathbf{R}$ is completely additive, then there exists $c \in \mathbf{R}$ so that $F(x) = cx$ for any $x \in [0, L]$.*

Proof. (i) Let $P := \{n \mid F(\lambda_n) > 0\}$ and $M := \mathbf{N} - P$. Putting $\xi := \sum_{n \in P} \lambda_n$ we see by (3.1) that

$$\sum_{n=1}^{\infty} |F(\lambda_n)| = \sum_{n \in P} F(\lambda_n) - \sum_{n \in M} F(\lambda_n) = F(\xi) - F(L - \xi) < \infty,$$

hence F is bounded in $[0, L]$.

(ii) Let $x, y \geqq 0$ and $x + y \leqq L$ be arbitrary. Then there exist sequences $\varepsilon_n, \delta_n \in \{0, 1\}$ such that $\varepsilon_n + \delta_n \in \{0, 1\}$ $(n \in \mathbf{N})$ and

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n, \quad y = \sum_{n=1}^{\infty} \delta_n \lambda_n.$$

By the complete additivity

$$F(x+y) = F\left[\sum_{n=1}^{\infty} (\varepsilon_n + \delta_n)\lambda_n\right] = \sum_{n=1}^{\infty} (\varepsilon_n + \delta_n) F(\lambda_n) =$$

$$= \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n) + \sum_{n=1}^{\infty} \delta_n F(\lambda_n) = F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) +$$

$$+ F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) = F(x) + F(y).$$

Now, as is known from [2], there exists an additive function $A : \mathbf{R} \to \mathbf{R}$ (i. e. $A(u+v) = A(u) + A(v)$ for any $u, v \in \mathbf{R}$) such that $A(x) = F(x)$ for $x \in [0, L]$. On the other hand, the boundedness of F implies that A is bounded in $[0, L]$, and from this $A(x) = cx$ follows (see e.g. [1]), i.e. we have $F(x) = cx$. □

4. The following definitions have also been introduced in [4].

Definition 4.1. Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence. The number $x \in [0, L]$ is said to be *uniquely determined,* if there exists a unique sequence $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$ for which (1.1) is satisfied.

Definition 4.2. Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence. The number $x \in]0, L[$ is said to be *finite* if there exist $N \in \mathbf{N}$ and $\varepsilon_n \in \{0, 1\}$ $(n = 1, 2, \dots, N-1)$ such that

$$x = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \lambda_N. \tag{4.1}$$

Remarks. (i) Clearly, 0 and L are always uniquely determined.

(ii) If $x \in [0, L]$ is finite, then it is not uniquely determined. Indeed, by Theorem 1.1, $\mu_n := \lambda_{N+n}$ $(n \in \mathbf{N})$ is in this case interval filling and $\lambda_N \leqq \sum_{i=N+1}^{\infty} \lambda_i = \sum_{n=1}^{\infty} \mu_n$. Thus there exists a sequence $\delta_n \in \{0, 1\}$ such that

$$\lambda_N = \sum_{n=1}^{\infty} \delta_n \mu_n = \sum_{i=N+1}^{\infty} \delta_{i-N} \lambda_i,$$

hence

$$x = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \lambda_N = \sum_{n=1}^{N-1} \varepsilon_n \lambda_n + \sum_{i=N+1}^{\infty} \delta_{i-N} \lambda_i,$$

i.e. x is not uniquely determined.

Theorem 4.1. *Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence for which it is true that any non-finite number from $]0, L[$ is uniquely determined. Then*

$$\lambda_n = \frac{2\lambda_1}{2^n} \quad (n \in \mathbf{N}).$$

Proof. (i) Suppose that there exists an $n \in \mathbf{N}$ for which

$$\lambda_n < \sum_{i=n+1}^{\infty} \lambda_i.$$

Let us show that in this case there exists a number in $]0, L[$ which is neither finite nor uniquely determined. The finite numbers form a countable set in $]0, L[$, so there exists $\eta > 0$ such that $\lambda_n + \eta$ is non-finite and

$$\lambda_n + \eta < \sum_{i=n+1}^{\infty} \lambda_i.$$

Now the non-finite number $x = \lambda_n + \eta$ has a representation in which λ_n occurs, and also such a one in which λ_n does not occur. Thus x is neither finite nor uniquely determined. From this, by Theorem 1.1, we obtain that

$$\lambda_n = \sum_{i=n+1}^{\infty} \lambda_i$$

for any $n \in \mathbf{N}$, hence

$$\lambda_n = \sum_{i=n+1}^{\infty} \lambda_i = \lambda_{n+1} + \sum_{i=n+2}^{\infty} \lambda_i = 2\lambda_{n+1}.$$

By induction we get (4.2).

(ii) On the other hand it is known that the sequence (4.2) satisfies the hypothesis of the theorem (see e.g. [5]). □

Remark. Theorem 4.1. can be regarded as a characterization of the sequence $\frac{c}{2^n}$ $(c > 0)$.

5. Let $1 < q \leq 2$ and $\lambda_n := \frac{1}{q^n}$ $(n \in \mathbf{N})$. Then $\left\{\frac{1}{q^n}\right\} \in \Lambda$ is an interval filling sequence, and we have shown in [4] that for $q \leq \frac{\sqrt{5}+1}{2}$ every number $x \in]0, L[\left(L = \frac{1}{q-1}\right)$ is non uniquely determined. By Theorem 4.1 we know that there always exists an $x\in\,]0, L[$ which is neither finite nor uniquely determined. Also, we have proved in [4] that for $\frac{\sqrt{5}+1}{2}<q\leqq 2$ there exist uniquely determined numbers.

Now we are going to investigate the following question: given a sequence $\varepsilon_n\in\{0,1\}$ $(n\in\mathbf{N})$ i.e. given an element $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots)\in\{0,1\}^{\mathbf{N}}$, for which values $1<q\leqq 2$ will the number

$$x=\sum_{n=1}^{\infty}\frac{\varepsilon_n}{q^n}\in\left[0,\frac{1}{q-1}\right] \tag{5.1}$$

be uniquely determined?

Definition 5.1. Let $\varepsilon\in\{0,1\}^{\mathbf{N}}$ and $n\in\mathbf{N}$. Denote by $E_n^+(\varepsilon)$ $(E_n^-(\varepsilon))$ the set of all those real numbers $1<q\leqq 2$ for which there does not exist a $\delta\in\{0,1\}^{\mathbf{N}}$ such that $\delta_i=\varepsilon_i$ for $i<n$, $\delta_n>\varepsilon_n$ $(\delta_n<\varepsilon_n)$ and

$$\sum_{i=1}^{\infty}\frac{\varepsilon_i}{q^i}=\sum_{i=1}^{\infty}\frac{\delta_i}{q_i}.$$

Let moreover $E_n(\varepsilon):=E_n^+(\varepsilon)\cap E_n^-(\varepsilon)$,

$$E^+(\varepsilon):=\bigcap_{n=1}^{\infty}E_n^+(\varepsilon),\quad E^-(\varepsilon):=\bigcap_{n=1}^{\infty}E_n^-(\varepsilon)$$

and

$$E(\varepsilon):=E^+(\varepsilon)\cap E^-(\varepsilon).$$

Remark. (i) It follows from the definition that for $q\in E(\varepsilon)$ the number (5.1) is uniquely determined. The set $E(\varepsilon)$ will accordingly be called the *unicity set* of the sequence $\varepsilon\in\{0,1\}^{\mathbf{N}}$.

(ii) We have $E_n^+(\varepsilon)=E_n^-(1-\varepsilon)$ for any $n\in\mathbf{N}$, where $1:=(1,1,1,\dots)\in\{0,1\}^{\mathbf{N}}$. Thus it often suffices to consider the set $E_n^+(\varepsilon)$.

Theorem 5.1. *The relation $q\in E_n^+(\varepsilon)$ holds if and only if either*

$$\varepsilon_n=1 \tag{1}$$

or else

$$\varepsilon_n=0\ \text{ and }\ \sum_{i=n+1}^{\infty}\frac{\varepsilon_i}{q^i}<\frac{1}{q^n}. \tag{2}$$

Proof. (i) Suppose that $q\notin E_n^+(\varepsilon)$ and $\varepsilon_n=0$. Then there exists a $\delta\in\{0,1\}^{\mathbf{N}}$ such that

$$\sum_{i=1}^{\infty}\frac{\varepsilon_i}{q^i}=\sum_{i=1}^{\infty}\frac{\delta_i}{q^i}$$

moreover $\varepsilon_i=\delta_i$ for $i<n$, $\delta_n=1$, $\varepsilon_n=0$ and so

$$\sum_{i=n+1}^{\infty}\frac{\varepsilon_i}{q^i}=\frac{1}{q^n}+\sum_{i=n+1}^{\infty}\frac{\delta_i}{q^i}\geqq\frac{1}{q^n}.$$

(ii) Conversely, if

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} \geqq \frac{1}{q^n}$$

then

$$0 \leqq \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} - \frac{1}{q^n} < \sum_{i=n+1}^{\infty} \frac{1}{q^i}$$

and so by the interval filling property there exists a sequence $\delta_i \in \{0, 1\}$ $(i \geqq n+1)$ such that

$$0 \leqq \sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} - \frac{1}{q^n} = \sum_{i=n+1}^{\infty} \frac{\delta_i}{q^i}.$$

Putting $\delta_i = \varepsilon_i$ for $i < n$, $\delta_n = 1$ we now get

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}. \quad \square$$

Theorem 5.2. *Let* $n \in \mathrm{N}$ *and* $\varepsilon \in \{0, 1\}^{\mathrm{N}}$ *be fixed. Then there exists* $q_0 = q_0(n; \varepsilon) \in [1, 2]$ *such that*

$$E_n^+(\varepsilon) = [q_0, 2].$$

Proof. The condition

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon_i}{q^i} < \frac{1}{q^n}. \tag{5.2}$$

occuring in Theorem 5.1 is equivalent to

$$f(q) := \sum_{i=1}^{\infty} \varepsilon_{i+n} \left(\frac{1}{q}\right)^i < 1.$$

As sum function of a power series f is continuous, and the coefficients being nonnegative it is a monotone increasing function of $\frac{1}{q}$. Thus there exists $q_0 \in [1, 2]$ such that for $q < q_0$ (5.2) is satisfied. $\square$

Theorem 5.1 enables us determine the unicity set $E(\varepsilon)$ for certain sequences $\varepsilon \in \{0, 1\}^{\mathrm{N}}$.

Theorem 5.3. *Let* $\varepsilon \in \{0, 1\}^{\mathrm{N}}$ *be defined by*

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \in \{1, k+1, 2k+1, 3k+1, \dots\} \\ 0 & \text{if } n \notin \{1, k+1, 2k+1, 3k+1, \dots\} \end{cases} \tag{5.3}$$

with $k \geqq 2$ *a fixed natural number. Then*

$$E(\varepsilon) = \{q \in]1, 2] \mid q^k - q^{k-1} - q^{k-2} - \dots - q - 1 > 0\}. \tag{5.4}$$

Proof. For $1<q\leqq 2$ we have

$$\sum_{i=1}^{\infty}\frac{\varepsilon_i}{q^i}=\frac{1}{q}\left(1+\frac{1}{q^k}+\frac{1}{q^{2k}}+\dots\right)=\frac{q^{k-1}}{q^k-1}.$$

By Theorem 5.1

(5.5) $$E_n^+(\varepsilon)=\,]1,2]\text{ for } n\in\{1,k+1,2k+1,\dots\}.$$

If $n\in\{2,3,\dots,k\}$ and $q\in E_n^+(\varepsilon)$ (in case $\varepsilon_n=0$) then by Theorem 5.1 it is necessary and sufficient that

$$\frac{1}{q^n}\geqq\frac{1}{q^k}>\sum_{i=k+1}^{\infty}\frac{\varepsilon_i}{q^i}=\frac{1}{q^{k+1}}+\frac{1}{q^{2k+1}}+\dots=\frac{1}{q^k}\frac{q^{k-1}}{q^k-1}$$

should hold. By the periodicity of the sequence ε we get

(5.6) $$E_n^+(\varepsilon)\supseteq\{q\in\,]1,\,2]\,|\,q^k-q^{k-1}-1>0\}$$

for $n\notin\{1,k+1,2k+1,\dots\}$; if $n=lk$ $(l\in\mathbf{N})$ then equality holds in (5.6). From equations (5.5) and (5.6) we obtain

(5.7) $$E^+(\varepsilon)=\{q\in\,]1,\,2]\,|\,q^k-q^{k-1}-1>0\}.$$

By the definition and by Theorem 5.1

(5.8) $$E_n^-(\varepsilon)=E_n^+(1-\varepsilon)=\,]1,2]\text{ for } n\notin\{1,k+1,2k+1,\dots\}.$$

If $n=1$ (in case $1-\varepsilon_n=0$) and $q\in E_1^-(\varepsilon)=E_1^+(1-\varepsilon)$, then by Theorem 5.1 it is necessary and sufficient that

$$\frac{1}{q}>\sum_{i=2}^{\infty}\frac{1-\varepsilon_i}{q^i}=\frac{1}{q-1}-\frac{q^{k-1}}{q^k-1}.$$

should hold, and this in turn is equivalent to the inequality

(5.9) $$q^k-q^{k-1}-q^{k-2}-\dots-q-1>0.$$

By the periodicity of the sequence $1-\varepsilon$ and by (5.9) we get

(5.10) $$E_n^-(\varepsilon)=E_n^+(1-\varepsilon)=\{q\in[1,2]\,|\,q^k-q^{k-1}-\dots-q-1>0\}.$$

for $n\in\{1,k+1,2k+1,\dots\}$. Thus by (5.8) and (5.10)

(5.11) $$E^-(\varepsilon)=\{q\in\,]1,\,2]\,|\,q^k-q^{k-1}-\dots-q-1>0\}.$$

By the definition (5.7) and (5.11) imply

$$E(\varepsilon)=E^+(\varepsilon)\cap E^-(\varepsilon)=\{q\in\,]1,2]\,|\,q^k-q^{k-1}-\dots-q-1>0\}$$

i.e. (5.4) is valid. □

Remark. From Theorem 5.3 we get as a special case $(k=2)$ that if $\varepsilon=(101010\dots)\in\{0,1\}^{\mathbf{N}}$ then

$$E(\varepsilon)=\{q\in[1,2]\,|\,q^2-q-1>0\}=\left]\frac{\sqrt{5}+1}{2},\,2\right].$$

This will say that in case $\frac{\sqrt{5}+1}{2} < q \leqq 2$ the number

$$x = \sum_{i=1}^{\infty} \frac{1}{q^{2i-1}} = \frac{q}{q^2-1} \in \left]0, \frac{1}{q-1}\right[$$

is always uniquely determined with respect to the interval filling sequence $\left\{\frac{1}{q^n}\right\} \in \Lambda$ (see [4]).

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