(0,2) LACUNARY INTERPOLATION WITH SPLINES OF DEGREE 6

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1. Introduction

In 1955, J. Surányi and P. Turán [5] commenced the study of (0,2) lacunary interpolation. Recently, A. Meir and A. Sharma [3], B. K. Swartz and R. S. Varga [6], S. Demko [1], A. K. Varma [7], and J. Prasad and A. K. Varma [4] considered special interpolation problems. More recently Th. Fawzy [2] presented a new method for studying this problem.

In this paper, we consider the (0,2) interpolation and we construct spline polynomial $S_\Delta(x)$ satisfying certain conditions. We also prove that, if $y \in \mathrm{Lip}_{M_0}\alpha$, $0<\alpha\leqq 1$ and $y\in C^6[0,1]$, then the approximation is $0(h^{6-i})$ in $y^{(i)}(x)$, $i = 0(1)5$ and is $0(h^\alpha)$ in $y^{(6)}(x)$.

Let $[0, 1]$ be a finite closed interval, and let $\Delta = \{x_k\}_0^n$ with $0 = x_0<x_1<\ldots<x_k<x_{k+1}<\ldots<x_n = 1$ be a partition into n subintervals, where the values of a function $y(x)$ and its second derivatives are associated with the knots. Then, we define the spline polynomial solving the (0, 2) interpolation problem as follows:

$$S_\Delta(x) = \begin{cases} S_0(x), & x_0\leq x\leq x_1, \\ S_k(x), & x_k\leq x\leq x_{k+1}, \quad k = 1(1)n-3, \\ S_{n-2}(x), & x_{n-2}\leq x\leq x_{n-1}, \\ S_{n-1}(x), & x_{n-1}\leq x\leq x_n, \end{cases}$$

and for each of these splines, we prove its existence, uniqueness and convergence.

2. Existence and uniqueness of the polynomial spline

In our case, we have $y\in C^6[0, 1]$ and we consider the partition

$$\Delta: 0 = x_0<x_1<\ldots<x_k<x_{k+1}<\ldots<x_n = 1,$$

where $x_{k+1}-x_k = h$, $k = 0(1)n-1$.

Theorem 2.1. *Given two sets of arbitrary values* $\{y(x_k)\}_{k=0}^{n}$ *and* $\{y''(x_k)\}_{k=0}^{n}$ *then there exists a unique spline polynomial* $S_\Delta(x)$ *interpolating* $y(x)$ *in the interval* $[0, 1]$ *such that*

(2.1) $S_\Delta(x)\in\pi_6$ *on each interval* $[x_k, x_{k+1}]$, $k = 0(1)n-1$, *where* π_6 *denotes the set of all polynomials of degree* ≤ 6,

(2.2) $S_\Delta^{(p)}(x_k) = y_\Delta^{(p)}$; $k = 0(1)n-1$, $p = 0,2$,

$$S_\Delta(x)\in C^{(0,2)}[0, 1],$$

(2.3)
$$\text{i.e., } S_k(x_{k+1}) = S_{k+1}(x_{k+1}),\ k = 0(1)n-1, \textit{and}$$
$$S''_k(x_{k+1}) = S''_{k+1}(x_{k+1}),\ k = 0(1)n-1,$$

(2.4)
$$S_\Delta(x)\equiv\begin{cases} S_0(x) & \text{for } x_0\leq x\leq x_1, \\ S_k(x) & \text{for } x_k\leq x\leq x_{k+1},\ k = 1(1)n-3, \\ S_{n-2}(x) & \text{for } x_{n-2}\leq x\leq x_{n-1}, \\ S_{n-1}(x) & \text{for } x_{n-1}\leq x\leq x_n \end{cases}$$

where

(2.5)
$$S_0(x) = y_0+b_1^{(0)}(x-x_0)+\frac{1}{2!}y_0''(x-x_0)^2+\frac{1}{3!}b_3^{(0)}(x-x_0)^3+ + \sum_{r=4}^{6}\frac{1}{r!}b_r^{(1)}(x-x_0)^r,$$

(2.6)
$$S_k(x) = y_k+b_1^{(k)}(x-x_k)+\frac{1}{2!}y_k''(x-x_k)^2 + \sum_{r=3}^{6}\frac{1}{r!}b_r^{(k)}(x-x_k)^r,$$

(2.7)
$$S_{n-2}(x) = y_{n-2}+S'_{n-3}(x_{n-2})(x-x_{n-2})+\frac{1}{2!}y''_{n-2}(x-x_{n-2})^2+ + \sum_{r=3}^{6}\frac{1}{r!}S_{n-3}^{(r)}(x_{n-2})(x-x_{n-2})^r,$$

and

(2.8)
$$S_{n-1}(x) = y_{n-1}+S'_{n-2}(x_{n-1})(x-x_{n-1})+\frac{1}{2!}y''_{n-1}(x-x_{n-1})^2+ + \sum_{r=3}^{6}\frac{1}{r!}S_{n-2}^{(r)}(x_{n-1})(x-x_{n-1})^r,$$

where $b_4^{(k)}$, $b_5^{(k)}$, *and* $b_6^{(k)}$ *are defined by*

(2.9)
$$b_4^{(k)} = \frac{1}{h^2}\{y''(x_{k+1})-2y''(x_k)+y''(x_{k-1})\},\quad k = 1(1)n-1,$$

(2.10)
$$b_5^{(k)} = \frac{1}{h^3}\{y''(x_{k+2}) - 3y''(x_{k+1}) + 3y''(x_k) - y''(x_{k-1})\},$$
$$k = 1(1)n-2,$$

and

(2.11)
$$b_6^{(k)} = \frac{1}{h^4}\{y''(x_{k+3}) - 4y''(x_{k+2}) + 6y''(x_{k+1}) -$$
$$-4y''(x_k) + y''(x_{k-1})\}, \quad k = 1(1)n-3,$$

and the other coefficients $b_1^{(0)}$, $b_3^{(0)}$, $b_1^{(k)}$, *and* $b_3^{(k)}$, $k = 1(1)n-3$ *are constants to be determined.*

Proof. Using (2.3), (2,9), (2.10), and (2.11), we get the following results:
(a) For the spline function $S_0(x)$, $x_0 \le x \le x_1$; •

(2.12)
$$b_3^{(0)} = \frac{1}{h}\left\{(y_1'' - y_0'' - \sum_{r=2}^{4} \frac{h^r}{r!} b_{r+2}^{(1)}\right\} \text{ and}$$

(2.13)
$$b_1^{(0)} = \frac{1}{h}\left\{(y_1 - y_0) - \frac{h^2}{2!}y_0'' - \frac{h^3}{3!}b_3^{(0)} - \sum_{r=4}^{6} \frac{h^r}{r} b_r^{(1)}\right\}.$$

(b) For the spline function $S_k(X)$, $x_k \le x \le x_{k+1}$, where $k = 1(1)n-3$,

(2.14)
$$b_3^{(k)} = \frac{1}{h}\left\{(y_{k+1}'' - y_k'') - \sum_{r=2}^{4} \frac{h^r}{r!} b_{r+2}^{(k)}\right\}$$

and

(2.15)
$$b_1^{(k)} = \frac{1}{h}\left\{(y_{k+1} - y_k) - \frac{h^2}{2!}y_k'' - \sum_{r=3}^{6} \frac{h^r}{r!} b_r^{(k)}\right\}.$$

Now, for the spline function $S_{n-2}(x)$, $x_{n-2} \le x \le x_{n-1}$, we use (2.6) for $k = n-3$ and then easily we get:

(2.16)
$$S_{n-3}'(x_{n-2}) = b_1^{(n-3)} + h_{n-3}''h + \sum_{r=3}^{6} \frac{h^{r-1}}{(r-1)!} b_r^{(n-3)},$$

(2.17)
$$S_{n-3}^{(r)}(x_{n-2}) = \sum_{i=r}^{6} \frac{h^{i-r}}{(i-r)!} b_i^{(n-3)},$$

where $r = 3(1)6$, thus the spline function $S_{n-1}(x)$ in uniquely determined.

Similarly, for the spline function $S_{n-1}(x)$, $x_{n-1} \le x \le x_n$, we use (2.7), (2.16), and (2.17), thus we have:

(2.18)
$$S_{n-2}'(x_{n-1}) = S_{n-3}'(x_{n-2}) + y_{n-2}''h + \sum_{r=3}^{6} \frac{h^{r-1}}{(r-1)!} S_{n-3}^{(r)}(x_{n-2})$$

(2.19)
$$S_{n-2}^{(r)}(x_{n-1}) = \sum_{i=r}^{6} \frac{h^{i-r}}{(i-r)!} S_{n-3}^{(i)}(x_{n-2}), \quad r = 3(1)6.$$

Thus, $S_\Delta(x)$ defined in (2.4) exists and unique. □

3. Convergence

Before proceeding to state and prove the convergence theorems, we need the following lemmas which will be used in the proof of convergence.

For simplicity, we use the following notations:

$$N = |Y^{(5)}(x_0)|, \; M = \|y^{(6)}(x)\|_\infty \text{ and } \omega_6(h)$$

is the usual modulus of continuity of $y^{(6)}(x)$.

Lemma 3.1. *Let $b_6^{(k)}$ be the constant given in* (2.11). *Then the inequality*

$$|b_6^{(k)} - y^{(6)}(x)| \le \frac{14}{3}\omega_6(h)$$

holds for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n-3$.

Proof. Using (2.11), the Taylor expansions of y''_{k+3}, y''_{k+2}, y''_{k+1}, y''_k and the definition of the modulus of continuity, Lemma (3.1) could easily be proved. □

Lemma 3.2. *Let $b_5^{(k)}$ be the constant given in* (2.10). *Then the inequality*

$$|b_5^{(k)} - y_k^{(5)}| \le m_1 h,$$

holds for all $k = 1(1)n-3$, where $m_1 = \frac{4}{3}M_0 + \frac{1}{2}M$, $y \in \mathrm{Lip}_{M_0}\alpha$ and $0 < \alpha \le 1$.

Proof. Using (2.10), the Taylor expansions of y''_{k+2}, y''_{k+1} and y''_k, and the definition of the modulus of continuity, we get:

$$|b_5^{(k)} - y_k^{(5)}| \le h\left\{\frac{4}{3}\omega_6(h) + \frac{1}{2}|y^{(6)}(\eta^{(k-1)})|\right\} \le h\left\{\frac{4}{3}M_0 + \frac{1}{2}M\right\} = m_1 h,$$

where $x_{k-1} < \eta^{(k-1)} < x_k$. Hence we get the proposition. □

Lemma 3.3. *Let $b_4^{(k)}$ be the constant given in* (2.9). *Then the inequality*

$$|b_4^{(k)} - y_k^{(4)}| \le m_2 h^2,$$

holds for all $k = 1(1)n-3$, where $m_2 = \frac{1}{3}M_0 + \frac{1}{12}M$, $y \in \mathrm{Lip}_{M_0}\alpha$ and $0 < \alpha \le 1$.

Proof. Using (2.9), the Taylor expansions of y''_{k+1} and y''_k, and the definition of the modulus of continuity, we could easily prove this lemma. □

Lemma 3.4. *Let $b_3^{(k)}$ be the constant given in* (2.14). *Then the inequality*

$$|b_k^{(k)} - y''_k| \le n_3 h^3,$$

holds for all $k = 1(1)n-3$, where $m_3 = \frac{7}{12}M_0 + \frac{1}{8}M$, $y \in \mathrm{Lip}_{M_0}\alpha$ and $0 < \alpha \le 1$.

Proof. By using (2.14), and the Taylor expansion of y''_{k+1}, we get:

$$|b_3^{(k)}-y_k'''| \leq \frac{h}{2}|b_4^{(k)}-y_k^{(4)}| + \frac{h^2}{3!}|b_5^{(k)}-y_k^{(5)}| + \frac{h^3}{4!}|b_6^{(k)}-y^{(6)}(\eta^{(k)})|,$$

where $x_k<\eta^{(k)}<x_{k+1}$.

Using Lemmas (3.1), (3.2), and (3.3), then Lemma (3.4) follows. □

Lemma 3.5. *Let $b_1^{(k)}$ be the constant given in* (2.15). *Then, the inequality*

$$|b_1^{(k)}-y_4'| \leq m_4 h^4$$

holds for all $k = 1(1)n-3$, where $m_4 = \frac{17}{216}M_0+\frac{19}{1440}M$, $y \in \mathrm{Lip}_{M_0}$ and $0<\alpha\leq 1$.

Proof. Using (2.15), the Taylor expansion of y_{k+1}, and Lemmas 3.1.–3.3, then Lemma 3.5 follows. □

Theorem 3.1. *Let $S_k(x)$ be the spline polynomial given in* (2.6). *If $y \in C^6[0, 1]$, then the inequalities*

$$|S_k^{(r)}(x)-y^{(r)}(x)| \leq C_{k,r}h^{6-r}, r = 0(1)5,$$

$$|S_k^{(6)}(x)-y^{(6)}(x)| \leq C_{k,6}\omega_6(h),$$

hold for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n-3$, where

$$C_{k,0} = \frac{28}{135}M_0+\frac{1}{24}M, \quad C_{k,1} = \frac{281}{540}M_0+\frac{53}{480}M,$$

$$C_{k,2} = \frac{7}{6}M_0+\frac{1}{4}M, \quad C_{k,3} = \frac{85}{36}M_0+\frac{11}{24}M, \quad C_{k,4} = 4M_0+\frac{7}{12}M,$$

$$C_{k,5} = 6M_0+\frac{1}{2}M \text{ and } C_{k,6} = \frac{14}{3}M_0.$$

Proof. Using (2.6) and the Taylor expansion of $y(x)$, we obtain for all $x \in [x_k, x_{k+1}]$ and $k = 1(1)n-3$,

$$|S_k(x)-y(x)| \leq h|b_1^{(k)}-y_k'| + \left\{\sum_{r=3}^{5}\frac{h^r}{6!}|b_r^{(k)}-y_k^{(r)}|\right\}+\frac{h^6}{6!}|b_6^{(k)}-y^{(6)}(x)|.$$

Using Lemmas 3.1–3.5 we get:

$$|S_k(x)-y(x)| \leq C_{k,0}h^6,$$

where $C_{k,0} = \frac{28}{135}M_0+\frac{1}{24}M$.

Similarly, using (2.6), Taylor expansions of $y^{(r)}(x)$; $r=1(1)5$ and Lemmas 3.1.–3.5 we get:

$$|S_k^{(r)}(x)-y^{(r)}(x)| \leq C_{k,r}h^{6-r}, \quad r = 1(1)5.$$

Finally, from (2.6) and Lemma 3.1 we get

$$|S_k^{(6)}(x)-y^{(6)}(x)| \leq \frac{14}{3}\,\omega_6(h),$$

and this completes the proof. □

Theorem 3.2. *Let $S_{n-2}(x)$ be the spline polynomial given in* (2.7). *If $y \in C^6\,[0, 1]$, then the inequalities*

$$|S_{n-2}^{(r)}(x)-y^{(r)}(x)| \leq C_{n-2,r}h^{6-r},\ r = 0(1)5,$$

$$|S_{n-2}^{(6)}(x)-y^{(6)}(x)| \leq C_{n-2,6}\,\omega_6(h),$$

hold for all $x \in [x_{n-2}, x_{n-1}]$, where

$$C_{n-2,0} = \frac{2459}{2160}M_0+\frac{31}{144}M,\ C_{n-2,1} = \frac{1439}{540}M_0+\frac{659}{1440}M,$$

$$C_{n-2,2} = \frac{403}{72}M_0+\frac{5}{6}M,\ C_{n-2,3} = \frac{371}{36}M_0+\frac{31}{24}M,$$

$$C_{n-2,4} = \frac{77}{6}M_0+\frac{13}{12}M,\ C_{n-2,5} = \frac{35}{3}M_0+\frac{1}{2}M,\ \textit{and}$$

$$C_{n-2,6} = \frac{17}{3}M_0.$$

To prove this theorem we state the following lemmas which could be easily proved by using Lemmas 3.1.—3.5, (2.16), and (2.17).

Lemma 3.6. *Let $S_{n-3}^{(6)}(x_{n-2}) = b_6^{(n-3)}$ be the constant given from* (2.11) *and* (2.17). *Then the inequality*

$$|S_{n-3}^{(6)}(x_{n-2})-y^{(6)}(x)| \leq \frac{17}{3}\,\omega_6(h)$$

holds for all $x \in [x_{n-2}, x_{n-1}]$.

Lemma 3.7. *Let $S_{n-3}^{(i)}(x_{n-2})$, $i = 1, 3, 4$ and 5 be the values given from* (2.16) *and* (2.17) *when $r = 3(1)5$. Then, the inequality*

$$|S_{n-3}^{(i)}(x_{n-2})-y_{n-2}^{(i)}| \leq a_i h^{6-i}$$

holds for all $i = 1, 3, 4$ and 5, where

$$a_1 = \frac{281}{540}M_0+\frac{53}{480}M,\ a_3 = \frac{85}{36}M_0+\frac{11}{24}M,\ a_4 = 4M_0+\frac{7}{12}M,$$

$$a_5 = 6\,M_0+\frac{7}{12}M,\ y \in \mathrm{Lip}_{M_0}\alpha\ \textit{and}\ 0<\alpha\leq 1.$$

Proof of Theorem 3.2. Using (2.7), the Taylor expansion of $y(x)$ for $x \in [x_{n-2}, x_{n-1}]$ and Lemmas 3.6., 3.7. we can easily complete the proof. □

Theorem 3.3. *Let $S_{n-1}(x)$ be the spline polynomial given in* (2.8). *If $y \in C^6[0, 1]$, then the inequalities*

$$|S_{n-1}^{(6)}(x) - y^{(6)}(x)| \le C_{n-1,r} h^{6-r}, \; r = 0(1)5,$$

$$|S_{n-1}^{(6)}(x) - y^{(6)}(x)| \le C_{n-1,6}\, \omega_6(h)$$

hold for all $x \in [x_{n-1}, x_n]$ where

$$C_{n-1,0} = \frac{3617}{720} M_0 + \frac{13}{18} M, \; C_{n-1,1} = \frac{5669}{540} M_0 + \frac{1879}{1440} M,$$

$$C_{n-1,2} = \frac{341}{18} M_0 + \frac{23}{12} M, \; C_{n-1,3} = \frac{361}{12} M_0 + \frac{21}{8} M,$$

$$C_{n-1,4} = \frac{167}{6} M_0 + \frac{4}{3} M, \; C_{n-1,5} = \frac{55}{3} M_0 + \frac{1}{2} M, \text{ and}$$

$$C_{n-1,6} = \frac{20}{3} M_0.$$

Before proving this theorem, we state some estimations in the following lemmas which could be easily proved using Lemma 3.6., Lemma 3.7., (2.18), and (2.19).

Lemma 3.8. *Let $S_{n-2}^{(6)}(x_{n-1})$ be the constant given from* (2.19) *when $r = 6$. Then, the inequality*

$$|S_{n-2}^{(6)}(x_{n-1}) - y^{(6)}(x)| \le \frac{20}{3} \omega_6(h)$$

holds for all $x \in [x_{n-1}, x_n]$.

Lemma 3.9. *Let $S_{n-2}^{(i)}(x_{n-1})$, $i = 1, 3, 4$ and 5 be the values given from* (2.18) *and* (2.19). *Then the inequality*

$$|S_{n-2}^{(i)}(x_{n-1}) - y_{n-1}^{(i)}| \le d_i h^{6-i}$$

holds for all $i = 1, 3, 4$ and 5, where

$$d_1 = \frac{1439}{540} M_0 + \frac{659}{1440} M, \; d_3 = \frac{371}{36} M_0 + \frac{31}{24} M, \; d_4 = \frac{77}{6} M_0 + \frac{13}{12} M,$$

$$d_5 = \frac{35}{3} M_0 + \frac{1}{2} M, \; y \in \mathrm{Lip}_{M_0}\alpha \text{ and } 0 < \alpha \le 1.$$

Proof of Theorem 3.3. Using (2.8), the Taylor expansion of $y(x)$ for $x \in [x_{n-1}, x_n]$ and Lemmas 3.8., 3.9., we can complete the proof. □

Finally, before, proving the convergence theorem concerning $S_0(x)$ for all $x \in [x_0, x_1]$, we present the following lemmas, which could be proved using the same techniques as in the previous lemmas.

Lemma 3.10. *Let* $b_6^{(1)}$ *be the constant given in* (2.11) *when* $k = 1$. *Then, the inequality*

$$|b_6^{(1)} - y^{(6)}(x)| \leq \frac{17}{3}\omega_6(h),$$

holds for all $x \in [x_0, x_1]$.

Lemma 3.11. *Let* $b_5^{(1)}$ *be the constant given in* (2.10) *when* $k = 1$. *Then the inequality*

$$|b_5^{(1)} - y_0^{(5)}| \leq m_5 h$$

holds, where $m_5 = \dfrac{4}{3} M_0 + \dfrac{3}{2} M$, $y \in \mathrm{Lip}_{M_0}\alpha$ *and* $0 < \alpha \leq 1$.

Lemma 3.12. *Let* $b_4^{(1)}$ *be the constant given in* (2.9) *when* $k = 1$. *Then the inequality*

$$|b_4^{(1)} - y_0^{(4)}| \leq m_6 h$$

holds, where $m_6 = \dfrac{1}{3} M_0 + \dfrac{7}{12} M + N$, $y \in \mathrm{Lip}_{M_0}\alpha$ *and* $0 < \alpha \leq 1$.

Lemma 3.13. *Let* $b_3^{(0)}$ *be the constant given in* (2.12). *Then the inequality*

$$|b_3^{(0)} - y_0'''| \leq m_7 h^2$$

holds, where $m_7 = \dfrac{5}{8} M_0 + \dfrac{13}{24} M + \dfrac{1}{2} N$, $y \in \mathrm{Lip}_{M_0}\alpha$ *and* $0 < \alpha \leq 1$.

Lemma 3.14. *Let* $b_1^{(0)}$ *be the constant given in* (2.13). *Then the inequality*

$$|b_1^{(0)} - y_0'| \leq m_8 h^4$$

holds, where $m_8 = \dfrac{37}{270} M_0 + \dfrac{61}{480} M + \dfrac{1}{8} N$, $y \in \mathrm{Lip}_{M_0}\alpha$ *and* $0 < \alpha \leq 1$.

Theorem 3.4. *Let* $S_0(x)$ *be the spline polynomial given in* (2.5). *If* $y \in C^6[0, 1]$, *then the inequalities*

$$|S_0^{(r)}(x) - y^{(r)}(x)| \leq C_{0,r} h^{5-r},\ r = 0(1)4,$$

$$|S_0^{(5)}(x) - y^{(5)}(x)| \leq C_{0,5} h,$$

$$|S_0^{(6)}(x) - y^{(6)}| \leq C_{0,6}\omega_6(h),$$

hold for all $x \in [x_0, x_1]$, *where*

$$C_{0,0} = \frac{37}{135}M_0 + \frac{61}{240}M + \frac{1}{4}N, \; C_{0,1} = \frac{1313}{2160}M_0 + \frac{803}{1440}M + \frac{13}{24}N,$$

$$C_{0,2} = \frac{5}{6}M_0 + \frac{13}{12}M + N, \; C_{0,3}\frac{185}{72}M_0 + \frac{15}{8}M + \frac{3}{2}N,$$

$$C_{0,4} = \frac{9}{2}M_0 + \frac{25}{12}M + N, \; C_{0,5} = 7M_0 + \frac{3}{2}M \text{ and } C_{0,6} = \frac{17}{6}M_0.$$

Proof. Using (2.5), Lemmas 3.10–3.14 and the Taylor expansion of $y(x)$, $x \in [x_0, x_1]$, we can easily complete the proof. □

The method described in this paper has been tested by the following example:

$$y(x) = 1 + xe^x, \; x \in [0, 1], \; x_k = kh, \; k = 0(1)10.$$

The results for $x = 0.55$ are given in the following table:

	Exact value	*Numerical value*	*Error*
y	1.953289160	1.953289210	$5(10)^{-8}$
$y^{(1)}$	2.68654178	2.686542251	$7.3(10)^{-8}$
$y^{(2)}$	4.419795196	4.419746589	$4.8607(10)^{-5}$
$y^{(3)}$	6.153048214	6.152790970	$2.57244(10)^{-4}$
$y^{(4)}$	7.886301232	7.925041325	$3.8740093(10)^{2}$
$y^{(5)}$	9.61955425	10.238529	$6.1897475(10)^{-1}$
$y^{(6)}$	11.35280727	12.0521	$6.9929273(10)^{-1}$

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