AN ALGORITHM FOR THE BEST CHEBYSHEV APPROXIMATIONS

FERENC KÁLOVICS
Mathematical Institute of Technical University for Heavy Industry
H-3515 Miskolc, Egyetemváros

(Received March 14, 1984)

The first part of the paper gives two theorems for characterization of the best Chebyshev approximations (the basic idea of Theorem 1.1 can be found in [1]). In the second part we shall give possibilities of the numerical application of Theorem 1.2.

1. On the characteristic properties of the best approximation

The fixed symbols are as follows: $[a, b]\subset\mathbf{R}$, a closed and bounded interval; $f(x)$, a continuous function on $[a, b]$; n denotes a natural number; $g_1(x), g_2(x), \ldots, g_n(x)$, a Chebyshev system of the continuous functions on $[a, b]$, i.e.

$$\begin{vmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_n(x_1) & \cdots & g_n(x_n) \end{vmatrix} > 0$$

if $a \le x_1 < x_2 < \ldots < x_n \le b$;

$A_1^*, \ldots, A_n^*$, the coefficients of the best Chebyshev approximation, i.e.

$$\max_{x\in[a,b]} |A_1^* g_1(x) + \ldots + A_n^* g_n(x) - f(x)| <$$

$$< \max_{x\in[a,b]} |A_1 g_1(x) + \ldots + A_n g_n(x) - f(x)|,$$

where $A^* \equiv \{A_1^*, \ldots, A_n^*\} \in \mathbf{R}^n$, $A \equiv \{A_1, \ldots, A_n\} \in \mathbf{R}^n$ and $A^* \ne A$;

$x_1^*, x_2^*, \ldots, x_{n+1}^*$, a set of the extremal points, i.e.

$$A_1^* g_1(x_i^*) + \ldots + A_n^* g_n(x_i^*) - f(x_i^*) =$$
$$= -\left(A_1^* g_1(x_{i+1}^*) + \ldots + A_n^* g_n(x_{i+1}^*) - f(x_{i+1}^*)\right) \quad (i = 1, \ldots, n)$$

and

$$|A_n^* g_1(x_i^*) + \ldots + A_n^* g_n(x_i^*) - f(x_i^*)| =$$
$$= \max_{x \in [a, b]} |A_1^* g_1(x) + \ldots + A_n^* g_n(x) - f(x)| \quad (i = 1, \ldots, n+1)$$

where a $\le x_1^* < x_2^* < \ldots < x_{n+1}^* \le b$.

Theorem 1.1. *The problem*

$$\begin{array}{l} a \le x_1 < x_2 < \ldots < x_{n+1} \le b \\ s_1 + s_2 + \ldots + s_{n+1} = 1 \\ s_1 g_1(x_1) - s_2 g_1(x_2) + \ldots + (-1)^n s_{n+1} g_1(x_{n+1}) = 0 \\ \vdots \\ s_1 g_n(x_1) - s_2 g_n(x_2) + \ldots + (-1)^n s_{n+1} g_n(x_{n+1}) = 0 \\ \hline |s_1 f(x_1) - s_2 f(x_2) + \ldots + (-1)^n s_{n+1} f(x_{n+1})| \to \max \end{array}$$

is solvable and $x_1^*, \ldots, x_{n+1}^*$ *is a set of the extremal points for each solution* $s_1^*, \ldots, s_{n+1}^*, x_1^*, \ldots, x_{n+1}^*$.

Proof. Since $g_1(x), \ldots, g_n(x)$ is a Chebyshev system, therefore the matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ g_1(x_1) & -g_1(x_2) & \ldots & (-1)^n g_1(x_{n+1}) \\ \vdots & \vdots & & \vdots \\ g_n(x_1) & -g_n(x_2) & \ldots & (-1)^n g_n(x_{n+1}) \end{bmatrix}$$

is nonsingular if $a \le x_1 < x_2 < \ldots < x_{n+1} \le b$. From the equations of the conditions we can see that

$$0 < s_i = \frac{\begin{vmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots \\ g_1(x_1) & \ldots & (-1)^{i-2} g_1(x_{i-1}) & 0 & (-1)^i g_1(x_{i+1}) & \ldots \\ \vdots & & \vdots & \vdots & \vdots & \end{vmatrix}}{\det G} < 1,$$

for all $a \leq x_1 < x_2 < \ldots < x_{n+1} \leq b$. Now let $s \equiv \{s_1, s_2, \ldots, s_{n+1}\}$ and $S \equiv \equiv \{S_1, S_2, \ldots, S_{n+1}\} = \{s_1, -s_2, \ldots, (-1)^n s_{n+1}\}$. If $s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}$ satisfies conditions of the problem, then

$$|s_1 f(x_1) - s_2 f(x_2) + \ldots + (-1)^n s_{n+1} f(x_{n+1})| = \left|\sum_{i=1}^{n+1} f(x_i) S_i\right| =$$

$$= \left|\sum_{i=1}^{n+1} (A_1^* g_1(x_i) + \ldots + A_n^* g_n(x_i) - f(x_i)) S_i\right| \leq$$

$$\leq \max_i |A_1^* g_1(x_i) + \ldots + A_n^* g_n(x_i) - f(x_i)| \sum_{i=1}^{n+1} |S_i| \leq$$

$$\leq \max_{x \in [a,b]} |A_1^* g_1(x) + \ldots + A_n^* g_n(x) - f(x)|.$$

Equalities are in places of the inequalities if $x_1, \ldots, x_n$ is a set of the extemal points. □

Theorem 1.2. *Assume that $f(x), g_1(x), \ldots, g_n(x)$ are continuously differentiable on $[a, b]$ and $a, b \notin X^*$, where X^* is a set of the extremal points. Then the elements $x_1^* < x_2^* < \ldots < x_{n+1}^*$ of X^* satisfy the system of equations*

$$g_1'(x_i) D_1(x_1, \ldots, x_{n+1}) + \ldots + g_n'(x_i) D_n(x_1, \ldots, x_{n+1}) =$$
$$= f'(x_i) D(x_1, \ldots, x_{n+1}), \quad (i = 1, 2, \ldots, n+1),$$

where

$$D(x_1, \ldots, x_{n+1}) = \begin{vmatrix} 1 & g_1(x_1) & \ldots g_n(x_1) \\ -1 & g_1(x_2) & \ldots g_n(x_2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (-1)^n & g_1(x_{n+1}) & \ldots g_n(x_{n+1}) \end{vmatrix}, D_1(x_1, \ldots, x_{n+1}) =$$

$$= \begin{vmatrix} 1 & f(x_1) & g_2(x_1) & \ldots g_n(x_1) \\ -1 & f(x_2) & g_2(x_2) & \ldots g_n(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^n & f(x_{n+1}) & g_2(x_{n+1}) & \ldots g_n(x_{n+1}) \end{vmatrix}, \ldots, D_n(x_1, \ldots, x_{n+1}) =$$

$$= \begin{vmatrix} 1 & g_1(x_1) & \ldots g_{n-1}(x_1) & f(x_1) \\ -1 & g_1(x_2) & \ldots g_{n-1}(x_2) & f(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^n & g_1(x_{n+1}) & \ldots g_{n-1}(x_{n+1}) & f(x_{n+1}) \end{vmatrix}.$$

Proof. If we determine $s_1^*, \ldots, s_{n+1}^*$ for the sequence $x_1^* < x_2^* < \ldots < x_{n+1}^*$ from the former formulas, then we get a solution of the problem of Theorem 1.1.

Since s_i^* and x_i^* are inner points of the intervals $[0, 1]$ and $[a, b]$ ($i = 1, \ldots, n+2$), respectively, therefore $s_1^*, \ldots, s_{n+1}^*, x_1^*, \ldots, x_{n+1}^*$ satisfy the Lagrange's conditions for

$$F(s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}) = s_1 f(x_1) - \ldots + (-1)^n s_{n+1} f(x_{n+1}) +$$
$$+ C_1(s_1 + \ldots s_{n+1} - 1) + C_2(s_1 g_1(x_1) - \ldots + (-1)^n s_{n+1} g_1(x_{n+1})) +$$
$$+ \ldots + C_{n+1}(s_1 g_n(x_1) - \ldots + (-1)^n s_{n+1} g_n(x_{n+1})), \text{ i.e.}$$
$$F'_{s_i} = 0, \; F'_{x_i} = 0 \; (i = 1, \ldots, n+1).$$

Hence

$$f(x_1) + C_1 + C_2 g_1(x_1) + \ldots + C_{n+1} g_n(x_1) = 0$$
$$-f(x_2) + C_1 - C_2 g_1(x_2) - \ldots - C_{n+1} g_n(x_2) = 0$$
$$\vdots$$
$$(-1)^n f(x_{n+1}) + C_1 + (-1)^n C_2 g_1(x_{n+1}) + \ldots + (-1)^n C_{n+1} g_n(x_{n+1}) = 0$$

and

$$s_1 f'(x_1) + s_1 C_2 g_1'(x_1) + \ldots + s_1 C_{n+1} g_n'(x_1) = 0$$
$$-s_2 f'(x_2) - s_2 C_2 g_1'(x_2) - \ldots - s_2 C_{n+1} g_n'(x_2) = 0$$
$$\vdots$$
$$(-1)^n s_{n+1} f'(x_{n+1}) + (-1)^n s_{n+1} C_2 g_1'(x_{n+1}) + \ldots +$$
$$+ (-1)^n s_{n+1} C_{n+1} g_n'(x_{n+1}) = 0.$$

If we determine $C_1, C_2, \ldots, C_{n+1}$ from the first system of the equations and we use these values in the second system, then we get the system of equations of our theorem. (Since $s_i \neq 0$, we can divide the equations of the second system by s_i!) □

Remark. If $a \in X^*$ or $b \in X^*$ or $a, b \in X_1^*$, then $x_1^* = a$ or $x_{n+1}^* = b$ or $x_1^* = a$, $x_{n+1}^* = b$, respectively.

Therefore from

$$F'_{s_i} = 0 \; (i = 1, \ldots, n+1), \; F'_{x_i} = 0 \; (i = 2, \ldots, n+1) \text{ or}$$
$$F'_{s_i} = 0 \; (i = 1. \ldots. n+1). \; F'_{x_i} = 0 \; (i = 1, \ldots, n) \text{ or}$$
$$F'_{s_i} = 0 \; (i = 1, \ldots, n+1), \; F'_{x_i} = 0 \; (i = 2, \ldots, n)$$

we get a result similar to Theorem 1.2.

The results in case $g_1(x) \equiv 1$, $g_2(x) \equiv x$:

(1) If $a, b \notin X^*$, then

$$\begin{vmatrix} 1 & 1 & f(x_1^*) \\ -1 & 1 & f(x_2^*) \\ 1 & 1 & f(x_3^*) \end{vmatrix} = f'(x_i^*) \begin{vmatrix} 1 & 1 & x_1^* \\ -1 & 1 & x_2^* \\ 1 & 1 & x_3^* \end{vmatrix} \Leftrightarrow f'(x_i^*) = \frac{f(x_3^*) - f(x_1^*)}{x_3^* - x_1^*}, \ (i=1,2,3).$$

(2) If $a \in X^*$ and $b \notin X^*$, then

$$\begin{vmatrix} 1 & 1 & f(a) \\ -1 & 1 & f(x_2^*) \\ 1 & 1 & f(x_3^*) \end{vmatrix} = f'(x_i^*) \begin{vmatrix} 1 & 1 & a \\ -1 & 1 & x_2^* \\ 1 & 1 & x_3^* \end{vmatrix} \Leftrightarrow f'(x_i^*) = \frac{f(x_3^*) - f(a)}{x_3^* - a}, \ (i = 2,3).$$

(3) If $a \notin X^*$ and $b \in X^*$, then

$$f'(x_i^*) = \frac{f(b) - f(x_1^*)}{b - x_1^*}, \ (i = 1,2).$$

(4) If $a, b \in X^*$, then

$$f'(x_2^*) = \frac{f(b) - f(a)}{b - a}.$$

These results give a simple geometrical information about the extremal points.

2. On numerical applications of Theorem 1.2

Assume that $a, b \notin X^*$ (the other cases are similar). We shall use Theorem 1.2 in two versions:

(1) If $f(x), g_1(x), \ldots, g_n(x)$ are twice-continuously differentiable and we have a suitable approach of sequence $x_1^* < x_2^* < \ldots < x_{n+1}^*$, then we can compute $x_1^*, \ldots, x_{n+1}^*$ by Newton's method ($A_1^*, \ldots, A_n^*$ is easily computed from $x_1^*, \ldots, x_{n+1}^*$). Here we apply the Newton's method to $g_1(x) \equiv 1$, $g_2(x) \equiv x$, as follows:

$$f'(x_i) \begin{vmatrix} 1 & 1 & x_1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & f(x_1) \\ -1 & 1 & f(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} = 0, \ (i = 1,2,3)$$

and the elements of the Jacobi matrix:

$$J_{11} = f''(x_1) \begin{vmatrix} 1 & 1 & x_1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} + f'(x_1) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 0 & 0 & f'(x_1) \\ -1 & 1 & f(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} =$$

$$= 2(x_3 - x_1) f''(x_1),$$

$$J_{12} = f'(x_1)\begin{vmatrix} 1 & 1 & x_1 \\ 0 & 0 & 1 \\ 1 & 1 & x_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & f(x_1) \\ 0 & 0 & f'(x_2) \\ 1 & 1 & f(x_3) \end{vmatrix} = 0,$$

$$J_{13} = 2(f'(x_1)-f'(x_3)), \qquad J_{21} = 2(f'(x_1)-f'(x_2)),$$

$$J_{22} = 2(x_3-x_1)f''(x_2), \qquad J_{23} = 2(f'(x_2)-f'(x_3)),$$

$$J_{31} = 2(f'(x_1)-f'(x_3)), \qquad J_{32} = 0,$$

$$J_{33} = 2(x_3-x_1)f''(x_3).$$

Hence

$$\begin{bmatrix} (x_3-x_1)f''(x_1) & 0 & f'(x_1)-f'(x_3) \\ f'(x_1)-f'(x_2) & (x_3-x_1)f''(x_2) & f'(x_2)-f'(x_3) \\ f'(x_1)-f'(x_3) & 0 & (x_3-x_1)f''(x_3) \end{bmatrix} \begin{bmatrix} \tilde{x}_1-x_1 \\ \tilde{x}_2-x_2 \\ \tilde{x}_3-x_3 \end{bmatrix} =$$

$$= \begin{bmatrix} f(x_3)-f(x_1)-(x_3-x_1)f'(x_1) \\ f(x_3)-f(x_1)-(x_3-x_1)f'(x_2) \\ f(x_3)-f(x_1)-(x_3-x_1)f'(x_3) \end{bmatrix},$$

where x_1, x_2, x_3 and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are "old and new approach" of x_1^*, x_2^*, x_3^*, respectively.

(2) If $f(x), g_1(x), \dots, g_n(x)$ are thrice-continuously differentiable, then we can look for the solution (solutions) of our system of equations in some intervals $a_1 \le x_1 \le b_1, \dots, a_{n+1} \le x_{n+1} \le b_{n+1}$ $(a < a_1 < b_1 < a_2 < \dots < a_{n+1} < b_{n+1} < b)$ of $\mathbf{R}^{n+1}$ by method of [2]. (This method does not require an initial approach of x_1^*, x_2^*, x_3^*.)

Numerical example. Let $[a, b] = [-1{,}5; 2]$,

$$f(x) = 3x^4-4x^3-12x^2, \; g_1(x) \equiv 1, \; g_2(x) \equiv x.$$

From the graph of $f(x)$ we can see that $a, b \notin X^*$ ($f(x)$ has local minima at (-1) and 2, local maximum at 0; furthermore $f(-1) = -5, f(0) = 0$, f(2) = $= -32$). If we use the Newton's method with initial approach $x_1 = -1$, $x_2 = 0, x_3 = 2$, then for the first approach of the unique solution:

$$\begin{bmatrix} 108 & 0 & 0 \\ 0 & -72 & 0 \\ 0 & 0 & 216 \end{bmatrix} \begin{bmatrix} \tilde{x}_1+1 \\ \tilde{x}_2 \\ \tilde{x}_3-2 \end{bmatrix} = \begin{bmatrix} -27 \\ -27 \\ -27 \end{bmatrix} \text{ and } \begin{aligned} \tilde{x}_1 &= -1{,}25, \\ \tilde{x}_2 &= 0{,}375, \\ \tilde{x}_3 &= 1{,}875. \end{aligned}$$

We can get from Kantorovich's theorem (used for the second approach) that $\|\tilde{x}^{(3)}-x^*\|_\infty<0{,}0007$, where $\tilde{x}^{(3)}\approx\{-1{,}2018;0{,}3325;1{,}8609\}$ is the third approach of the exact solution $x^* = (x_1^*, x_2^*, x_3^*)$.

REFERENCES

[1] *Glashoff K.* and *Gustafson S. A.:* Linear Optimization and Approximation. Springer-Verlag, New York/Heidelberg/Berlin, 1983.

[2] *Erdélyi Z.* and *Kálovics F.:* On numerical applications of excluding theorems. *Annales Univ. Sci. Budapest. Sectio Computatorica* 4 (1983), 11–20.

[3] *Rice J. R.:* The Approximation of Functions, Vol. I: Linear Theory. Addison-Wesley Publ. Comp., London, 1964.