

**HERMITE INTERPOLATION AND THE TWO
POINT BOUNDARY VALUE PROBLEMS.
APPLICATION OF THE METHODS**

By

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(Received January 22, 1981)

The Hermite interpolation methods established in [2] and [3] for the solution of non-linear ordinary differential equations are applied in this paper to solve the two boundary value problems. Two examples are given and programs are written for them in FORTRAN IV. The programs are run in the computer ODR – 1304 (compatible with the ICL – 1900 series) giving approximately 11 digits of accuracy. Numerical results are obtained for comparison between the numerical exact solutions and the approximate ones.

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MASTER DIFFEQ
INTEGER NN(2)
REAL F(10)
REAL X(21),Y(21),YD(21),YDD(21)
DATA IDB/2/,NN/10,20/
CALL GENFAKT(F,10)
DO 1 I=1,IDB
N=NN(I)+1
CALL GENXNU(X,N)
Y (1)=0.0
YD (1)=-1.0+COT(1.0)
YDD(1)=-1.0
DO 2 I=2,N/2+1
H=X(I)-X(I-1)
H2=H*H/2.0
H3=H2*H/3.0
H4=H3*H/4.0
H5=H4*H/5.0
H6=H5*H/6.0
H7=H6*H/7.0
H8=H7*H/8.0
H9=H8*H/9.0
H10=H9*H/10.0
YDD(I)=Y (I-1)*(-1.0+H2-H4+H6)+
1 YD (I-1)*(-H+H3-H5+H7)+
2 YDD(I-1)*H8+
3 X (I-1)*(H6-H4+H2)+
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4          (H7 - H5 + H3 - H) - X(I - 1)
YD (I) = Y (I - 1) * (-H + H3 - H5 + H7) +
1          YD (I - 1) * (1.0 - H2 + H4 - H6 + H8) +
2          YDD(I - 1) * H9 +
3          X (I - 1) * (H7 - H5 + H3 - H) ÷
4          (H8 - H6 + H4 - H2)
Y (I) = Y (I - 1) * (1.0 - H2 + H4 - H6 + H8) +
1          YD (I - 1) * (H - H3 + H5 - H7 + H9) +
2          YDD(I - 1) * H10 +
3          X (I - 1) * (H8 - H6 + H4 - H2) +
4          (H9 - H7 + H5 - H3)
2 CONTINUE
WRITE (2,100) NN(I1)
100 FORMAT (1H1,25X, 'EXAMPLE 1.'/1H0,25X, 'N =',I3/1H0,5X,
1'NU X(NU) EXACT SOLUTION NUM. SOLUTION'/1H0)
DO 3 I=1,N/2+1
CAL EXACT1(X(I),E,ED,EDD)
NU=I-2
WRITE (2,101) NU,X(I),E,Y(I),ED,YD(I),EDD,YDD(I)
101 FORMAT (1H0,17,F13.7 ,4X,3HY =,F15.10,3X,3HY =,F15.10/
1          1X,24X.3HY ' =,F15.10,3X,3HY ' =,F15.10/
2          1X,24X.3HY" =,F15.10,3X,3HY" =,F15.10)
3 CONTINUE
DO 4 I=1,N/2
4 Y(N+1-I) = -Y(I+1)
WRITE(2,300) NN(I1)
300 FORMAT(1H1,25X,'EXAMPLE 1.'/1H0,25X,'N =',I3/1H0,8X,
1'X EXACT SOLUTION NUM. SOLUTION'/H0)
XX = -1.0
DO 5 I=1,9
SE = SIN(XX)/SIN(1.0) - XX
S = HEXI(XX,Y,X,N)
WRITE(2,102) XX,SE,S
102 FORMAT (1H0,F15.7,7X,F15.10,6X,F15.10)
5 XX = XX + 0.25
1 CONTINUE
DO 10 I1=1,IDB
N = NN(I1) + 1
CALL GENXNU(X,N)
Y (1) = 0.0
YD (1) = -(EXP(1.0) - EXP(-1.0))/(EXP(1.0) + EXP(-1.0))
YDD(1) = -1.0
DO 20 I=2,N/2+1
H = X(I) - X(I-1)
H2 = H * H/2.0
H3 = H2 * H/3.0
H4 = H3 * H/4.0
H5 = H4 * H/5.0
H6 = H5 * H/6.0
H7 = H6 * H/7.0
H8 = H7 * H/8.0
H9 = H8 * H/9.0
H10 = H9 * H/10.0
YDD(I) = Y (I - 1) * (1.0 + H2 + H4 + H6) +
1          YD (I - 1) * (H + H3 + H5 + H7) +
2          YDD(I - 1) * H8 -
3          (1.0 + H2 + H4 + H6)
YD (I) = Y (I - 1) * (H + H3 + H5 + H7) +

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1      YD (I-1)*(1.0+H2+H4+H6+H8)+
2      YDD(I-1)*H9-
3      (H+H3+H5+H7)
Y (I)=Y (I-1)*(1.0+H2+H4+H6+H8)+
1      YD (I-1)*(H+H3+H5+H7+H9)+
2      YDD(I-1)*H10-
3      (H2+H4+H6+H8)
20 CONTINUE
WRITE (2,200) NN(I1)
2 0 0 FORMAT (1H1,25X,'EXAMPLE 2.'/1H0,25X,'N=',13/1H0,5X,
1'NU X(NU) EXACT SOLUTION NUM.SOLUTION'/1H0)
DO 30 I=1,N/2+1
CALL EXACT2(X(I),E,ED,EDD)
NU=I-2
WRITE (2,101) NU,X(I),E,Y(I),ED,YD(I),EDD,YDD(I)
30 CONTINUE
DO 40 I=1,N/2
40 Y(N+1-I)=Y(I+1)
WRITE(2,400) NN(I1)
4 0 0 FORMAT (1H1,25X,'EXAMPLE 2.'/1H0,25X,'N=',13/1H0,8X,
1'X EXACT SOLUTION NUM. SOLUTION'/1H0)
XX=-1.0
DO 50 I=1,9
SE=1.0-(EXP(XX)+EXP(-XX))/(EXP(1.0)+EXP(-1.0))
S=HEX2(XX,Y,X,N)
WRITE(2,102) XX,SE,S
50 XX=XX+0.25
10 CONTINUE
STOP
END

SUBROUTINE GENFAKT(FAKT,N)
REAL FAKT (N)
FAKT(1)=1.0
DO 1 I=2,N
1 FAKT(I)=FAKT(I-1)*I
RETURN
END

SUBROUTINE GENXNU(X,N)
REAL X(N)
M=N+N-2
X(1)=1.0
PI=ACOS(-1.0)
DO 1 I=2,N
1 X(I)=COS((I+I-3)*PI/M)
RETURN
END

SUBROUTINE EXACT1(X,Y,YD,YDD)
S=SIN(1.0)
Y =SIN(X)/S-X
YD =COS(X)/S-1.0
YDD=-SIN(X)/S
RETURN
END

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SUBROUTINE EXACT2(X,Y,YD,YDD)
E=EXP(X)
F=EXP(-X)
C=1.0/(EXP(1.0)+EXP(-1.0))
Y=1.0-(E+F)*C
YD=-C*(E-F)
YDD=-C*(E+F)
RETURN
END

FUNCTION HEX1(X,Y,XNU,N)
REAL Y(N),XNU(N)
C=COS((N-1)*ACOS(X))
R0=((X-1.0)+0.5*(X-1.0)**2)*0.5*(1.0+X)*C
RN=((X+1.0)-0.5*(X+1.0)**2)*0.5*(1.0-X)*C
HEX1=(-1.0+COT(1.0))*(R0+RN)
S=0.0
DO 1 I=2,N
1 S=S+Y(I)*(-1.0)**I/((X-XNU(I))*(1.0-XNU(I)**2)**1.5)
HEX1=HEX1+(1.0-X*X)**2*C/(N-1)*S
RETURN
END

FUNCTION HEX2(X,Y,XNU,N)
REAL Y(N),XNU(N)
C=COS((N-1)*ACOS(X))
R0=((X-1.0)+0.5*(X-1.0)**2)*0.5*(1.0+X)*C
RN=((X+1.0)-0.5*(X+1.0)**2)*0.5*(1.0-X)*C
E=EXP(1.0)
E=E*E
HEX2=(1.0-E)/(1.0+E)*(R0-RN)
S=0.0
DO 1 I=2,N
1 S=S+Y(I)*(-1.0)**I/((X-XNU(I))*(1.0-XNU(I)**2)**1.5)
HEX2=HEX2+(1.0-X*X)**2*C/(N-1)*S
RETURN
END

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The program

Example 1

Consider the following differential equation (see [4], p. 212. for the problem itself, but the numerical method here is different and the results are better)

$$(1) \quad y'' + y + x = 0 \quad -1 \leq x \leq 1,$$

with the boundary conditions

$$(2) \quad \begin{aligned} y(-1) &= y(1) = 0 \\ y'(-1) &= y'(1) = -1 + \operatorname{ctg} 1. \end{aligned}$$

The exact solution is obviously

$$(3) \quad y = \frac{\sin x}{\sin 1} - x.$$

Equations (1.13–14) and (3.10–11) presented in [2] give the approximate values \bar{y}_{v+1} of the exact values y_{v+1} at the points

$$(4) \quad x_{v+1} = \cos \frac{2v+1}{2n} \pi \quad \left(0 \leq v \leq \left[\frac{n}{2} \right] - 1 \right).$$

In this example, we have from the asymmetry

$$(5) \quad y_{n-v} = -\dot{y}_{v+1} \quad \left(\left[\frac{n}{2} \right] \leq v \leq n-1 \right).$$

From (3.9–14) and (3.8) in [2], we obtain $\bar{y}_{-1} = y(-1) = 0$, $\bar{y}'_{-1} = -1 + \text{ctg } 1$ and

$$(6) \quad \bar{y}_{v+1} = A_{11}\bar{y}_v + B_{11}\bar{y}'_v + \frac{(x_{v+1}-x_v)^8}{8!}\bar{y}''_v + C_{11}x_v + D_{11} \\ \left(-1 \leq v \leq \left[\frac{n}{2} \right] - 1, x_{-1} = -1 \right)$$

where

$$(7) \quad A_{11} = -1 + \frac{(x_{v+1}-x)^2}{2!} - \frac{(x_{v+1}-x)^4}{4!} + \frac{(x_{v+1}-x)^6}{6!}, \\ B_{11} = -\frac{x_{v+1}-x_v}{1!} + \frac{(x_{v+1}-x_v)^3}{3!} - \frac{(x_{v+1}-x_v)^5}{5!} + \frac{(x_{v+1}-x_v)^7}{7!}, \\ C_{11} = A_1 + 1, \\ D_{11} = B_{11} - x_v.$$

$$(8) \quad \bar{y}'_{v+1} = A_{12}\bar{y}_v + B_{12}\bar{y}'_v + \bar{y}''_v \frac{(x_{v+1}-x_v)^9}{9!} + A_{12}x_v + D_{12},$$

where

$$(9) \quad A_{12} = B_{11}, \\ B_{12} = -A_{11} + \frac{(x_{v+1}-x_v)^8}{8!}, \\ D_{12} = B_{12} - 1.$$

And

$$(10) \quad \bar{y}_{v+1} = A_{13}\bar{y}_v + B_{13}\bar{y}'_v + \bar{y}''_v \frac{(x_{v+1}-x_v)^{10}}{10!} + C_{13}x_v + D_{13},$$

where

$$(11) \quad A_{13} = -A_{11} + \frac{(x_{v+1}-x_v)^8}{8!}, \\ B_{13} = -B_{11} + \frac{(x_{v+1}-x_v)^9}{9!}, \\ C_{13} = D_{12}, \\ D_{13} = \frac{(x_{v+1}-x_v)^9}{9!} - \frac{(x_{v+1}-x_v)^7}{7!} + \frac{(x_{v+1}-x_v)^5}{5!} - \frac{(x_{v+1}-x_v)^3}{3!}.$$

The approximate solution of (1-2) is thus given as follows

$$(12) \quad \bar{H}(x) = (-1 + \text{ctg } 1)\{r_{0,1}(x) + r_{n+1,1}(x)\} + \frac{1}{n} \sum_{\nu=0}^{n-1} \bar{y}_{\nu+1} \frac{(1-x^2)^2 [\cos(n \text{ arc } \cos x)] (-1)^\nu}{(x-x_{\nu+1})(1-x_{\nu+1}^2)^{3/2}}$$

where

$$(13) \quad r_{0,1}(x) = \left\{ (x-1) + \frac{1}{2} (x-1)^2 \right\} \left\{ \frac{1+x}{2} \right\} \cos(n \text{ arc } \cos x),$$

$$r_{n+1,1}(x) = \left\{ (x+1) - \frac{1}{2} (x+1)^2 \right\} \left\{ \frac{1-x}{2} \right\} \cos(n \text{ arc } \cos x).$$

In the equations (4-12), we let $n = 10$ and the program 1 is set for them. This program is run and the numerical results obtained are set out in tables 1.1-2.

To see the effect of n we repeated the procedure for $n = 20$ and the numerical results obtained are given in tables 1.3-4.

EXAMPLE 1.

N=10

X	EXACT SOLUTION	NUM. SOLUTION
-1.0000000	0.0000000000	0.0000000000
-0.7500000	-0.0600561663	-0.0600561664
-0.5000000	-0.0697469637	-0.0697469634
-0.2500000	-0.0440136543	-0.0440136546
0.0000000	0.0000000000	0.0000000003
0.2500000	0.0440136543	0.0440136542
0.5000000	0.0697469637	0.0697469638
0.7500000	0.0600561663	0.0600561663
1.0000000	0.0000000000	0.0000000000

Table 1.1.

EXAMPLE 1.

N=10

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
-1	1.0000000	Y = 0.0000000000 Y' = -0.3579073840 Y'' = -1.0000000000	Y = 0.0000000000 Y' = -0.3579073841 Y'' = -1.0000000000

0	0.9876883	Y = 0.0043308460 Y' = -0.3456446982 Y'' = -0.9920191866	Y = 0.0043308460 Y' = -0.3456446983 Y'' = -0.9920191866
1	0.8910065	Y = 0.0332141397 Y' = -0.2529396999 Y'' = -0.9242206639	Y = 0.0332141397 Y' = -0.2529397000 Y'' = -0.9242206639
2	0.7071068	Y = 0.0649185778 Y' = -0.0965290416 Y'' = -0.7720253589	Y = 0.0649185777 Y' = -0.0965290416 Y'' = -0.7720253589
3	0.4539905	Y = 0.0671864668 Y' = 0.0680156820 Y'' = -0.5211769666	Y = 0.0671864668 Y' = 0.0680156820 Y'' = -0.5211769666
4	0.1564345	Y = 0.0287141744 Y' = 0.1738836862 Y'' = -0.1851486395	Y = 0.0287141744 Y' = 0.1738836862 Y'' = -0.1851486396

Table 1.2.

EXAMPLE 1.

N = 20

X	EXACT SOLUTION	NUM. SOLUTION
-1.0000000	0.0000000000	0.0000000000
-0.7500000	-0.0600561663	-0.0600561665
-0.5000000	-0.0697469637	-0.0697469637
-0.2500000	-0.0440136543	-0.0440136542
0.0000000	0.0000000000	0.0000000000
0.2500000	0.0440136543	0.0440136545
0.5000000	0.0697469637	0.0697469634
0.7500000	0.0600561663	0.0600561662
1.0000000	0.0000000000	0.0000000000

Table 1.3.

EXAMPLE 1.

N = 20

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
-1	1.0000000	Y = 0.0000000000 Y' = -0.3579073840 Y'' = -1.0000000000	Y = 0.0000000000 Y' = -0.3579073841 Y'' = -1.0000000000
0	0.9969173	Y = 0.0010985608 Y' = -0.3548277735 Y'' = -0.9980158945	Y = 0.0010985607 Y' = -0.3548277735 Y'' = -0.9980158945

1	0.9723699	Y = 0.0095095804 Y' = -0.3305258978 Y'' = -0.9818795008	Y = 0.0095095804 Y' = -0.3305258979 Y'' = -0.9818795008
2	0.9238795	Y = 0.0243955005 Y' = -0.2837197551 Y'' = -0.9482750330	Y = 0.0243955005 Y' = -0.2837197552 Y'' = -0.9482750330
3	0.8526402	Y = 0.0422454138 Y' = -0.2180391764 Y'' = -0.8948855781	Y = 0.0422454138 Y' = -0.2180391765 Y'' = -0.8948855781
4	0.7604060	Y = 0.0586545359 Y' = -0.1389408718 Y'' = -0.8190605015	Y = 0.0586545359 Y' = -0.1389408718 Y'' = -0.8190605015
5	0.6494480	Y = 0.0692302243 Y' = -0.0535410902 Y'' = -0.7186782726	Y = 0.0692302243 Y' = -0.0535410902 Y'' = -0.7186782726
6	0.5224986	Y = 0.0705663134 Y' = 0.0298334710 Y'' = -0.5930648781	Y = 0.0705663134 Y' = 0.0298334709 Y'' = -0.5930648781
7	0.3826834	Y = 0.0610765385 Y' = 0.1024336786 Y'' = -0.4437599709	Y = 0.0610765385 Y' = 0.1024336786 Y'' = -0.4437599709
8	0.2334454	Y = 0.0414670253 Y' = 0.1561600693 Y'' = -0.2749123892	Y = 0.0414670253 Y' = 0.1561600692 Y'' = -0.2749123892
9	0.0784591	Y = 0.0146856771 Y' = 0.1847392029 Y'' = -0.0931447728	Y = 0.0146856771 Y' = 0.1847392028 Y'' = -0.0931447728

Table 1.4.

Example 2

Consider the differential equation (see [4] for the problem, but a different method is used and the numerical results are here considerably better).

$$(14) \quad y'' - y + 1 = 0$$

with the boundary conditions

$$(15) \quad \begin{aligned} y(-1) &= y(1) = 0 \\ y'(-1) &= \frac{e^2 - 1}{1 + e^2}, \quad y'(1) = \frac{1 - e^2}{1 + e^2}, \end{aligned}$$

The exact solution is obviously

$$(16) \quad y = 1 - \frac{\operatorname{ch} x}{\operatorname{ch} 1}.$$

The approximate values \bar{y}_{v+1} are evaluated at

$$(17) \quad x_{v+1} = \cos \frac{2v+1}{2n} \pi \quad \left(0 \leq v \leq \left[\frac{n}{2} \right] - 1 \right).$$

In this example, we have

$$(18) \quad \bar{y}_{n-r} = y_{v+1} \quad \left(\left[\frac{n}{2} \right] \leq v \leq n-1 \right).$$

From (3.8) and (3.9–14) presented in [2], we obtain

$$y_{-1} = y(-1) = 0, \quad \bar{y}'_{-1} = y'(-1) = \frac{e^2 - 1}{1 + e^2}$$

and

$$(19) \quad \bar{y}_{v+1} = A_{21} \bar{y}_v + B_{21} \bar{y}'_v + \bar{y}''_v \frac{(x_{v+1} - x_v)^8}{8!} - A_{21},$$

$$\left(-1 \leq v \leq \left[\frac{n}{2} \right] - 1, \quad x_{-1} = -1 \right)$$

where

$$(20) \quad A_{21} = 1 + \frac{(x_{v+1} - x_v)^2}{2!} + \frac{(x_{v+1} - x_v)^4}{4!} + \frac{(x_{v+1} - x_v)^6}{6!},$$

$$B_{21} = \frac{(x_{v+1} - x_v)}{1!} + \frac{(x_{v+1} - x_v)^3}{3!} + \frac{(x_{v+1} - x_v)^5}{5!} + \frac{(x_{v+1} - x_v)^7}{7!}.$$

$$(21) \quad \bar{y}'_{v+1} = A_{22} \bar{y}'_v + B_{22} \bar{y}''_v + \bar{y}'''_v \frac{(x_{v+1} - x_v)^9}{9!} - A_{22}$$

where

$$(22) \quad A_{22} = B_{21}$$

$$B_{22} = A_{21} + \frac{(x_{v+1} - x_v)^8}{8!}.$$

And

$$(23) \quad \bar{y}_{v+1} = A_{23} \bar{y}_v + B_{23} \bar{y}'_v + \bar{y}''_v \frac{(x_{v+1} - x_v)^{10}}{10!} + C_{23}$$

where

$$(24) \quad A_{23} = B_{22}$$

$$B_{23} = A_{22} + \frac{(x_{v+1} - x_v)^9}{9!}$$

$$C_{23} = -B_{22} + 1.$$

The approximate solution of (14 – 15) is thus given as follows

$$(25) \quad \begin{aligned} \bar{H}(x) = & \frac{1 - e^2}{1 + e^2} \{r_{0,1}(x) - r_{n+1,1}(x)\} + \\ & + \frac{1}{n} \sum_{\nu=0}^{n-1} \bar{y}_{\nu+1} \frac{(1 - x^2)^2 (\cos(n \arccos x)) (-1)^\nu}{(x - x_{\nu+1})(1 - x_{\nu+1}^2)^{3/2}}, \end{aligned}$$

where

$$(26) \quad \begin{aligned} r_{0,1}(x) &= \left\{ (x - 1) + \frac{1}{2}(x - 1)^2 \right\} \left\{ \frac{1 + x}{2} \right\} (\cos(n \arccos x)), \\ r_{n+1,1}(x) &= \left\{ (x + 1) - \frac{1}{2}(x + 1)^2 \right\} \left\{ \frac{1 - x}{2} \right\} (\cos(n \arccos x)). \end{aligned}$$

In the equations (17 – 26) we let $n = 10$ and we set for them program 2. The program is run and the numerical results obtained are given in tables 2.1 – 2. To see the effect of n , we repeated the procedure for $n = 20$ and the numerical results obtained are given in tables 2.3 – 4.

EXAMPLE 2.

N = 10

X	EXACT SOLUTION	NUM. SOLUTION
-1.0000000	0.0000000000	0.0000000000
-0.7500000	0.1609749643	0.1609749644
-0.5000000	0.2692371742	0.2692371732
-0.2500000	0.3315883327	0.3315883332
0.0000000	0.3519457263	0.3519457256
0.2500000	0.3315883327	0.3315883333
0.5000000	0.2692371742	0.2692371742
0.7500000	0.1609749643	0.1609749644
1.0000000	0.0000000000	0.0000000000

Table 2.1.

EXAMPLE 2.

N = 10

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
-1	1.0000000	Y = 0.0000000000 Y' = -0.7615941560 Y'' = -1.0000000000	Y = 0.0000000000 Y' = -0.7615941559 Y'' = -1.0000000000
0	0.9876883	Y = 0.0093009353 Y' = -0.7493399063 Y'' = -0.9906990647	Y = 0.0093009353 Y' = -0.7493399063 Y'' = -0.9906990647

1	0.8910065	Y = 0.0772275720 Y' = -0.6569129411 Y'' = -0.9227724280	Y = 0.0772275720 Y' = -0.6569129411 Y'' = -0.9227724280
2	0.7071068	Y = 0.1830680730 Y' = -0.4973966543 Y'' = -0.8169319270	Y = 0.1830680730 Y' = -0.4973966543 Y'' = -0.8169319270
3	0.4539905	Y = 0.2840063716 Y' = -0.3044216390 Y'' = -0.7159936284	Y = 0.2840063716 Y' = -0.3044216389 Y'' = -0.7159936284
4	0.1564345	Y = 0.3440000340 Y' = -0.1017920127 Y'' = -0.6559999660	Y = 0.3440000339 Y' = -0.1017920127 Y'' = -0.6559999661

Table 2.2.

EXAMPLE 2.

N = 20

X	EXACT SOLUTION	NUM. SOLUTION
-1.0000000	0.0000000000	0.0000000000
-0.7500000	0.1609749643	0.1609749648
-0.5000000	0.2692371742	0.2692371743
-0.2500000	0.3315883327	0.3315883317
0.0000000	0.3519457263	0.3519457274
0.2500000	0.3315883327	0.3315883343
0.5000000	0.2692371742	0.2692371731
0.7500000	0.1609749643	0.1609749640
1.0000000	0.0000000000	0.0000000000

Table 2.3.

EXAMPLE 2.

N = 20

NU	X(NU)	EXACT SOLUTION	NUM. SOLUTION
-1	1.0000000	Y = 0.0000000000 Y' = -0.7615941560 Y'' = -1.0000000000	Y = 0.0000000000 Y' = -0.7615941559 Y'' = -1.0000000000
0	0.9969173	Y = 0.0023429929 Y' = -0.7585151035 Y'' = -0.9976570071	Y = 0.0023429929 Y' = -0.7585151035 Y'' = -0.9976570071
1	0.9723699	Y = 0.0206638498 Y' = -0.7342512877 Y'' = -0.9793361502	Y = 0.0206638498 Y' = -0.7342512877 Y'' = -0.9793361502

2	0.9238795	Y = 0.0551303431 Y' = -0.6876076839 Y'' = -0.9448696569	Y = 0.0551303431 Y' = -0.6876076839 Y'' = -0.9448696569
3	0.8526402	Y = 0.1017578804 Y' = -0.6219843758 Y'' = -0.8982421196	Y = 0.1017578804 Y' = -0.6219843758 Y'' = -0.8982421196
4	0.7604060	Y = 0.1553840371 Y' = -0.5416657486 Y'' = -0.8446159629	Y = 0.1553840371 Y' = -0.5416657486 Y'' = -0.8446159629
5	0.6494480	Y = 0.2104048939 Y' = -0.4510943249 Y'' = -0.7895951061	Y = 0.2104048939 Y' = -0.4510943249 Y'' = -0.7895951061
6	0.5224986	Y = 0.2614538534 Y' = -0.3542260141 Y'' = -0.7385461466	Y = 0.2614538534 Y' = -0.3542260140 Y'' = -0.7385461466
7	0.3826834	Y = 0.3039111095 Y' = -0.2540972292 Y'' = -0.6960888905	Y = 0.3039111095 Y' = -0.2540972292 Y'' = -0.6960888905
8	0.2334454	Y = 0.3342069647 Y' = -0.1526631070 Y'' = -0.6657930353	Y = 0.3342069647 Y' = -0.1526631070 Y'' = -0.6657930353
9	0.0784591	Y = 0.3499500470 Y' = -0.0508979347 Y'' = -0.6500499530	Y = 0.3499500470 Y' = -0.0508979347 Y'' = -0.6500499530

Table 2.4.

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