THE RANDOM SECRETARY PROBLEM WITH MULTIPLE CHOICE

by

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1. Introduction

The best known and most frequently quoted optimal stopping problem is, without doubt, the secretary problem (also known as dowry, beauty contest or best choice problem), conceived as follows: n rankable candidates applying for a job appear in random order. The object is to select the best (rank 1), with maximal probability, under the following constraints:

- It must be decided immediately on arrival whether the candidate is to be accepted or rejected, as later recall is not permitted.
- The only information that can be utilized in the selecting procedure is the sequence of relative ranks with respect to the preceding arrivals.

This classical drafting appeared first in the Fifties or the early Sixties, see [4, 10], but a problem closely related to the above can be traced back to Cayley, 1875. In that ancient problem the observations came from a known distribution, thus the observed value contained further information in addition to the relative rank. This version is called the full information case while the classical secretary problem is referred to as the no information case.

In the last two decades several versions, artful generalizations of the original problem were considered. A recent survey paper [3] tries to give a true cross-section of all related investigations, on the basis of more than 60 references. In fact, the number of relevant papers comes to 100, and one can hardly ask a pertinent question that has not yet been invented.

Here we mention only two branches of the bush grown from the seeds of the classical secretary problem. A natural idea is to allow more than one trial for selecting the best. In their paper [6], Gilbert and Mosteller solved this problem in an intuitive way. The optimal stopping rule they obtained was given by a sequence of threshold indices $n \geqq k_1^* \geqq k_2^* \geqq \ldots$. In the case of r choices, the i^{th} selection is made at the arrival of the first leader (relative rank 1) after time k_{r+1-i}^* and after the $(i-1)$st selection, of course. Following this stopping rule the limit of the probability of selecting the best candidate (success) can be arbitrarily close to 1 as $n \to \infty$, if r is large enough.

The same problem was solved by Sakaguchi [21] using the method described in [2], i.e. dealing with the Markov chain formed by the arrival times of successive leaders.

Henke [8] and Nikolaev [14] aimed at minimizing the expected rank sum of the chosen individuals. In [15, 20, 22, 23] two choices are allowed in selecting the two best candidates. In [24], a slight modification of the above problem is investigated: the object is to select one of the two best individuals. Müller and Platen solved the general problem of selecting the m best candidates [13, 14].

Motivated in particular by problems studied by Gilbert and Mosteller [6], Haggstrom worked out the theoretical background of optimal sequential procedures in the case of more than one stop [7].

In a recent paper [11], the limit behaviour of the optimal stopping rule is investigated in the case of several choices with an uncertainty of selection.

Another direction of generalizations is to randomize the number of applicants. The first paper in this connection was that of Presman and Sonin [17]. They let the number of candidates be a random variable with known distribution $\{p_i\}$. As they have proved, the optimal stopping rule selects the first leader whose arrival time belongs to Γ, a subset of the positive integers, which depends on the distribution $\{p_i\}$. They also determined a class of distributions for which the characteristic set Γ was rather simple.

Gianini-Pettitt minimized the expected rank of the selected candidate for certain distributions [5]. Petruccelli studied the problem in the case of a general utility function [16]. The earlier papers [18, 19] concerning this case were in part erroneous. In [9], Irle revisited the original problem of Presman and Sonin. He applied Howard's policy iteration method, based on a work of Rasche.

The aim of the present paper is to join these two directions by giving a common extension. In this respect, we refer to [25] where a special case is studied.

2. Results

We consider the probabilistic model described in [17]. We also accept the notations used there. Instead of repeating the construction line by line, we recall it with some explanatory words.

Let the number of candidates be denoted by ι, a positive integer valued random variable with distribution $P(\iota = i) = p_i$, $i = 1, 2, \ldots$. Suppose all the $k!$ orderings of the applicants are equally probable under the condition $\iota = k$. Let y_k denote the relative rank of the k^{th} arrival, i.e. if $\iota \geqq k$, let $y_k - 1$ be the number of earlier arrived persons better than the one under examination, otherwise let $y_k = +\infty$ by definition. Denote by ξ the arrival time of the best candidate. We may have r choices, these being represented by the stopping times $\tau_1 \leqq \tau_2 \leqq \ldots \leqq \tau_r$ with respect to the increasing sequence of σ-fields generated by the random variables $y_1, y_2, \ldots$. Here $\tau_l = +\infty$ is allowed, indicating that less than l choices are made, but $\tau_{l-1} < \tau_l$ is required if the latter is finite. We aim at determining

$$\sup P(\tau_i = \xi \text{ for some } i = 1, 2, \ldots, r)$$

where the supremum is taken over all r-tuplets of stopping times. We are also interested in describing the optimal stopping rule.

We confine our attention to distributions for which

$$(1) \qquad \frac{1}{p_i} \sum_{j=i+1}^{\infty} \frac{p_j}{j} \quad \text{is non-increasing}$$

(here $\frac{0}{0} = 0$ and for $x > 0$ $\frac{x}{0} = +\infty$ by definition).

This condition is shown to be satisfied in some important special cases (uniform, Poisson, geometric distribution), see [17]. It is easy to prove that each discrete IFR distribution, i.e. for which

$$p_i \Big/ \sum_{j=i}^{\infty} p_j \quad \text{is non-increasing,}$$

satisfies condition (1). A distribution with the property

$$p_i^2 \geqq p_{i-1}\, p_{i+1}$$

is called doubly positive or discrete logarithmic concave. These distributions constitute a proper subclass of the discrete IFR distributions. The three examples considered by Presman and Sonin all belong to this family.

Let ξ_i denote the arrival time of the i^{th} leader and let $\xi_i = +\infty$ if less than i leaders are seen. That is, $\xi_1 = 1$ and

$$\xi_i = \inf\{m : m > \xi_{i-1}, y_m = 1\}.$$

Theorem 1. *Suppose that the distribution of ι satisfies condition* (1). *Then the optimal stopping rule is similar to the one obtained by Gilbert and Mosteller, i.e. there exists a sequence of threshold indices $k_1^* \geqq k_2^* \geqq \ldots \geqq 1$ not depending on r such that*

$$\tau_l = \inf\{\xi_i : \xi_i > \tau_{l-1},\ \xi_i \geqq k^*_{r+1-l}\}, \qquad l = 1, 2, \ldots, r$$

(where $\tau_0 = 0$). □

In words, the l^{th} choice should fall on the first leader arriving at or after the time k^*_{r+1-l}.

Remark 1. The optimal strategy is not necessarily unique: it can happen that more than one choice of the threshold indices gives the same success probability. In this case the thresholds can be chosen from certain intervals, independently of each other, but paying attention to the right order.

In the sequel we study the limit properties of the threshold sequence and the success probability as the number of candidates tends stochastically to infinity. Let the distribution of ι be a function of a positive parameter λ.

Suppose that $\lambda^{-1}\iota_\lambda$ converges, as $\lambda\to\infty$, to some $\iota\not\equiv 0$ in distribution. The following theorem deals with the properties of the limit distribution. Though we shall not apply it in full, the result may be of independent interest.

Theorem 2. *a) The distribution of* ι *is either degenerate or it is absolutely continuous inside* $H=(0, \operatorname{sup\,ess}\iota)$.

b) The corresponding density function is of form

$$f(x) = Cr(x)\exp\left(-\int^{x}\frac{r(s)}{s}\,ds\right),\quad x\in H, \tag{2}$$

where r is a nonnegative increasing function on H, with $r(+0)<1$.

c) f is continuous in H except for countable many points which are discontinuities of the first kind, at these points f is locally increasing, i.e. $f(x-0)\le f(x)\le f(x+0)$, *further, f is differentiable a.e.*

d) $\lim\limits_{\lambda\to\infty}\lambda p_{[\lambda x]}=f(x)$ *if x belongs to the continuity set of f* ([.] *stands for integer part*). □

Remark 2. As will be seen from the proof,

$$r(x) = f(x)\Big/\int_x^\infty\frac{f(s)}{s}\,ds.$$

In the sequel, we assume that the set where $r(x)=1$ is either void or it contains not more than one point. This condition assures the uniqueness of the optimal stopping rule, at least in the limit.

Denote by $P_r^*(\lambda)$ the maximal probability of success achieved by r choices and write $k_i^*(\lambda)$, $i=1,2,\ldots$, for the threshold sequence of the optimal stopping rule in order to indicate the dependence on λ.

Theorem 3.

a)
$$\lim_{\lambda\to\infty}\lambda^{-1}k_i^*(\lambda)=x_i^*$$

where $\{x_i^*\}$ *is a strictly decreasing sequence of positive real numbers tending to* 0.

b) The numbers x_k^* *are determined by the recursion*

$$I_1(x_k)=\sum_{j=1}^{k-1}\frac{1}{(k+1-j)!}E\left(\frac{1}{\iota}(\ln\iota)^{k+1-j}\,\mathbf{I}({}^*_j<\iota\le x^*_{j-1})\right)-\\ -\sum_{j=1}^{k-1}\frac{1}{(k+1-j)!}(\ln x_j^*)^{k+1-j}E\left(\frac{1}{\iota}\mathbf{I}(\iota>x_j^*)\right) \tag{3}$$

where

$$I_1(x)=E\left[\frac{1}{\iota}\left(1+\ln\frac{x}{\iota}\right)\mathbf{I}(\iota>x)\right]$$

and $\mathbf{I}(.)$ *denotes the indicator of the event in.* □

Remark 3. A more compact form of the defining relation of the sequence $\{x_k^*\}$ is the following identity

$$\sum_{j=1}^{\infty} t^{j-1} E\left(\frac{1}{\iota} \mathbf{I}(\iota > x_j^*)\left(x_j^{*\iota} - (1-t)\iota^t\right)\right) \equiv 0, \quad |t| < 1. \tag{4}$$

Especially, in the case of degenerate limit distribution $\iota \equiv 1$, we obtain an identity known from [11]

$$\sum_{j=1}^{\infty} t^{j-1} x_j^{*t} \equiv 1, \quad |t| < 1.$$

Theorem 4.

a) $$P_r^* = \lim_{\lambda \to \infty} P_r^*(\lambda) = \sum_{j=1}^{r} E\left(\frac{x_j^*}{\iota} \mathbf{I}(\iota > x_j^*)\right).$$

b) $$\lim_{r \to \infty} P_r^* = P(\iota > 0). \quad \square$$

Remark 4. If we allow $r(x)$ to take the value 1 on a complete interval, then it is not sure that equation (3) can be solved uniquely and part a) of Theorem 3 should be replaced by the following assertion:

Every limit point $(x_1^*, x_2^*, \ldots)$ of the sequence $\lambda^{-1}(k_1^*(\lambda), k_2^*(\lambda), \ldots)$ satisfies (3). The converse is not necessarily true, but for every $(x_1^*, x_2^*, \ldots)$ satisfying (3) and for every $\varepsilon > 0$, there exists a sequence of ε-optimal stopping rules of the type described in Theorem 1 along which

$$\lim_{\lambda \to \infty} \lambda^{-1} k_i^*(\lambda) = x_i^*, \quad i = 1, 2, \ldots.$$

3. Proofs

Proof of Theorem 1. We apply Dynkin's method of Markov chains. Our problem can easily be reduced to the problem of optimal stopping the special Markov chain ξ_i, $i = 1, 2, \ldots$, where the reward function $g_l(k)$ is defined as the conditional probability of success, when the first choice is made at the k^{th} arrival, supposing he is a leader. This reduction is described in [17] for $l = 1$ and it can be done in the same way for $l > 1$.

Denote by $s_l(k)$, the conditional probability of selecting the best when no choice is made before the k^{th} arrival, supposing again that he is a leader. Then

$$g_1(k) = k \sum_{j=k}^{\infty} \frac{p_j}{j} \Big/ \sum_{j=k}^{\infty} p_j$$

and

$$g_l(k) = g_1(k) + \sum_{j=k+1}^{\infty} p(k|j) s_{l-1}(j), \tag{5}$$

where $p(k|j)$ denotes the transition probability

$$P(\xi_l = j | \xi_{l-1} = k) = \frac{k \sum_{\nu=j}^{\infty} p_\nu}{j(j-1) \sum_{\nu=j}^{\infty} p_\nu}, \quad j > k.$$

Further,

$$s_l(k) = \max\left\{g_l(k), \sum_{j=k+1}^{\infty} p(k|j) s_l(j)\right\} \tag{6}$$

and $s_l(1)$ gives the success probability.

The characteristic set of the optimal policy is $\Gamma_l = \{k : s_l(k) = g_l(k)\}$. More precisely, if the two terms on the right hand side of (6) are equal for a positive integer k, then Γ_l is not unique in the sense that k may be left out. In order to clear up the structure of Γ_l, we apply Theorem 3.1 and the Lemma of [17].

First we have to introduce some new notations,

$$G_l(k) = \frac{1}{k} g_l(k) \sum_{j=k}^{\infty} p_j,$$

$$S_l(k) = \frac{1}{k} s_l(k) \sum_{j=k}^{\infty} p_j.$$

Defining the operator T as

$$Tf(k) = \sum_{j=k}^{\infty} \frac{f(j+1)}{j},$$

we can rewrite (5) and (6) as

$$\begin{aligned} G_l(k) &= G_1(k) + TS_{l-1}(k) \\ S_l(k) &= \max\{G_l(k), TS_l(k)\}. \end{aligned} \tag{7}$$

Let $c_{l,k} = G_l(k) - TG_l(k)$, $c_k = c_{1,k}$. In virtue of the quoted results of Presman and Sonin, a sufficient condition for Γ_l to be of form $[k_l^*, \infty)$ is that the sequence $c_{l,k}$, $k = 1, 2, \dots$ be negative and increasing if $k < k_l^*$ and nonnegative if $k \geqq k_l^*$. (If $c_{l,k} = 0$ for $k \in [k_l^*, m-1]$ and $c_{l,m} > 0$, then we can choose $\Gamma_l = [k, +\infty)$ for any $k \in [k_l^*, m]$, as well.

We show that this condition is fulfilled, with $k_1^* \geqq k_2^* \geqq \dots$. Since $\lim_{k\to\infty} c_{l,k} = 0$ and by (7)

$$\begin{aligned} c_{l,k} - c_{l,k+1} &= G_l(k) - G_l(k+1) - TG_l(k) + TG_l(k+1) = \\ &= G_l(k) - \frac{k+1}{k} G_l(k+1) = \\ &= G_1(k) - \frac{k+1}{k} G_1(k+1) + TS_{l-1}(k) - \frac{k+1}{k} TS_{l-1}(k+1) = \\ &= (c_k - c_{k+1}) + \frac{1}{k}\left(S_{l-1}(k) - TS_{l-1}(k+1)\right) = \\ &= (c_k - c_{k+1}) + \frac{1}{k} c_{l-1,k+1}^+, \end{aligned}$$

we obtain the recursion

$$c_{l,k} = c_k + Tc_{l-1,k}^{+}$$

where $x^{+} = \max\{x, 0\}$.

From hypothesis (1), it follows that the sequence

$$c_k = \sum_{i=k}^{\infty} \frac{p_i}{i}\left(1 - \frac{1}{p_i}\cdot\sum_{j=i+1}^{\infty}\frac{p_j}{j}\right) \tag{8}$$

possesses the required property (for this expression of c_k see (3.6) of [17]). Now the proof is inductive over the number of choices. Suppose we have carried out the proof in the case of $l-1$ choices. Then for $k \leqq k_{l-1}^{*}$ c_k increases and $Tc_{l-1,k}^{+}$ does not vary, thus $c_{l,k}$ increases as well. Clearly, $c_{l,k} \geqq c_{l-1,k}$, hence $c_{l,k_{l-1}^{*}} \geqq 0$, consequently $k_l^{*} \leqq k_{l-1}^{*}$ and the proof is complete. □

Proof of Theorem 2. From (1), it follows that for $i \leqq j \leqq k \leqq l$

$$p_k E\left(\frac{1}{\iota_\lambda}\mathbf{I}(\iota_\lambda > i)\right) \geqq p_j E\left(\frac{1}{\iota_\lambda}\mathbf{I}(\iota_\lambda > l)\right),$$

thus

$$\frac{P(k < \iota_\lambda \leqq k+r)}{rE\left(\frac{1}{\iota_\lambda}\mathbf{I}(\iota_\lambda > k+r)\right)} \geqq \frac{P(j < \iota_\lambda \leqq j+d)}{dE\left(\frac{1}{\iota_\lambda}\mathbf{I}(\iota_\lambda > j)\right)} \tag{9}$$

if $j+d \leqq k+r$ and $j \leqq l$. Hence we obtain

$$\frac{P(y < \iota \leqq y+\varrho)}{\varrho E\left(\frac{1}{\iota}\mathbf{I}(\iota > y+\varrho)\right)} \geqq \frac{P(x < \iota \leqq x+\delta)}{\delta E\left(\frac{1}{\iota}\mathbf{I}(\iota > x)\right)}.$$

for $0 < x \leqq y$, $x+\delta \leqq y+\varrho \in H$.

This implies a) immediately, further f is locally bounded in H, and with the notation $\Phi(x) = \int_x^{\infty} \frac{1}{s} f(s)ds$ the quotient $f(x)/\Phi(x) = r(x)$ is an increasing function of x. From this fact (2) follows by easy computation. Obviously

$$f(x) \geqq Cr(+0)\exp\left(-\int^{x}\frac{r(+0)}{s}ds\right) = C'x^{-r(+0)},$$

thus $r(+0) < 1$ by the integrability of f.

Part c) is a simple consequence of b).

Finally, using (9), one can see that for arbitrary $x \in H$ $\lambda p_{[\lambda x]}$ remains bounded as $\lambda \to \infty$, thus any subsequence of λ contains a further subsequence along which $\lambda p_{[\lambda x]}$ converges for every $x \in \mathbf{Q} \cap H$. Let us denote the limit by $g(x)$, further let $\underline{g}(x) = \liminf \lambda p_{[\lambda x]}$, $\bar{g}(x) = \limsup \lambda p_{[\lambda x]}$, where λ runs over this latter subsequence. From (9) it follows that the difference quotients of g and $\bar{g}$ are bounded from below in every compact subinterval of H, hence both g and $\bar{g}$ satisfy c. Since $g = \bar{g}$ in a dense subset of H, we have $g = \bar{g}$ at every common continuity point. Thus $\lambda p_{[\lambda x]}$ converges to a limit $g(x)$ at every continuity point $x \in H$, along the underlying subsequence of λ. Using Theorem 7.8 of [1] and the tightness of the sequence $\lambda^{-1}\iota_\lambda$ we obtain $f = g$ a. e., and, again by property c), $f = g$ at the continuity points. □

Proof of Theorem 3. *a)* We deal first with the case of one choice. In a sense our assertion is related to Theorem 4.1 of [17], wich states convergence with a rate λ^{-1} under conditions of another type. Consider the function $I_1(x, \lambda) = \lambda c_{[\lambda x]}$, $x > 0$. Then $k_1^*(\lambda) = \inf\{x > 0\colon I_1(x, \lambda) \geqq 0\}$. According to (4.1) of [17], we can rewrite $I_1(x, \lambda)$ in the form

$$I_1(x, \lambda) = \lambda \sum_{i=[\lambda x]}^{\infty} \frac{p_i}{i}\left(1 - \sum_{j=[\lambda x]}^{i-1} \frac{1}{j}\right) = $$
$$= E\left[\frac{\lambda}{\iota_\lambda}\left(1 - \sum_{j=[\lambda x]}^{\iota_\lambda - 1} \frac{1}{j}\right) \mathbf{I}(\iota_\lambda \geqq [\lambda x])\right].$$

Applying Theorem 5.5 of [1], one can see that

$$\lim_{\lambda \to \infty} I_1(x, \lambda) = E\left(\frac{1}{\iota}\left(1 - \ln\frac{\iota}{x}\right) \mathbf{I}(\iota > x)\right) = I_1(x) \tag{10}$$

if x is a ι-continuity point, i.e. $P(\iota = x) = 0$, thus (10) holds for every $x \in H$. Moreover, the convergence is uniform in every compact subset of H.

By an easy computation, we obtain

$$dI_1(x) = E\left(\frac{1}{\iota}\mathbf{I}(\iota > x)\right)\frac{dx}{x} - \frac{1}{x}\,dF(x),$$

where F denotes the distribution function of ι. Hence $I_1(x)$ is absolutely continuous in H with derivative

$$\frac{dI_1(x)}{dx} = \begin{cases} \dfrac{f(x)}{x}\left(\dfrac{1}{r(x)} - 1\right) & \text{if } r(x) > 0 \\[2ex] E\left(\dfrac{1}{\iota}\right)\dfrac{1}{x} & \text{if } r(x) = 0. \end{cases}$$

Next we show that $\int_H I_1(x)\,dx = 0$. For $0 < a < b$

$$\begin{aligned} \int_a^b I_1(x)dx &= E\left[\int_a^b \frac{1}{\iota}\left(1 + \ln\frac{x}{\iota}\right)\mathbf{I}(\iota > x)dx\right] = \\ &= E\left(\frac{b}{\iota}\ln\frac{b}{\iota}\mathbf{I}(\iota > b)\right) - E\left(\frac{a}{\iota}\ln\frac{a}{\iota}\mathbf{I}(\iota > a)\right) \end{aligned} \tag{11}$$

which tends to 0 as $a \to 0$ and $b \to \infty$, by the dominated convergence theorem.

These results together imply that the equation $I_1(x) = 0$ has a unique solution in H; denote it by x_1^*. By the monotony of $I_1(x, \lambda)$ when being negative, it follows that $\lambda^{-1}k_1^*(\lambda) \to x_1^*$.

Passing over to the case of more choices, let us define the functions $I_l(x)$, $l = 2, 3, \ldots$, recursively.

$$I_l(x) = I_1(x) + \int_x^\infty \frac{1}{y} I_{l-1}^+(y)dy, \quad x > 0. \tag{12}$$

We show by induction over the number of choices that $I_l(x)$ is well defined and it has a unique root x_l^* in H. Similarly to the case of one choice, one can prove that $I_l(x, \lambda) = \lambda c_{l,[\lambda x]}$ converges to $I_l(x)$ uniformly in every compact subset of H, further $\lim\limits_{\lambda\to\infty} \lambda^{-1}k_l^*(\lambda) = x_l^*$. First, in virtue of (11)

$$\lim_{b\to 0} \frac{1}{b} \int_0^b I_1(x)dx = \lim_{b\to 0} E\left(\frac{1}{\iota} \ln \frac{b}{\iota} \mathbf{I}(\iota > b)\right) = -\infty$$

by the monotone convergence theorem, hence $\lim\limits_{x\to +0} I_1(x) = -\infty$ Keeping in mind the monotony of I_1 in $(0, x_1^*)$, we have to prove that

$$\int_0^\infty \frac{1}{x} I_l^+(x)dx < \infty, \quad l = 1, 2, \ldots .$$

Starting from the supposition for l, we can write

$$\int_0^\infty \frac{1}{x} I_l^+(x)dx \leq \frac{1}{x_l^*} \int_0^\infty I_l^+(x)dx$$

and

$$\int_0^\infty I_l^+(x)dx \leq \int_0^\infty I_l^+(x)dx + \int_0^\infty \int_x^\infty \frac{1}{y} I_{l-1}^+(y)dydx =$$

$$= \int_0^\infty I_1^+(x)dx + \int_0^\infty \int_0^y \frac{1}{y} I_{l-1}^+(y)dxdy =$$

$$= \int_0^\infty I_1^+(x)dx + \int_0^\infty I_{l-1}^+(x)dx \leq \ldots \leq l \int_0^\infty I_1^+(x)dx.$$

Thus I_{l+1} is also well defined and obviously $x_{l+1}^* < x_l^*$, since $I_l(x) < I_{l+1}(x)$. Finally, $x_l^* \to 0$ will follow from part *a)* of Theorem 4.

b) Let $x_k^* \leqq x \leqq x_{k-1}^*$ (where $x_0^* = \text{sup ess}\, \iota$) and

$$T_k(t, x) = \sum_{l=k}^{\infty} t^{l-k} I_l(x).$$

Then

$$\frac{\partial}{\partial x} T_k(t, x) = \frac{1}{1-t} I_1'(x) - \frac{t}{x} T_k(t, x).$$

Solving this differential equation, we get

$$T_k(t, x) = x^{-t}\left(\varkappa_k(t) - \frac{1}{1-t}\int_x^{x_0^*} y^t I_1'(y)dy\right) =$$

$$= x^{-t}\left(\varkappa_k(t) - \frac{1}{t} E\left(\iota^{t-1}\mathbf{I}(\iota > x)\right)\right) + \frac{1}{t(1-t)} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x)\right).$$

The functions $\varkappa_k(t)$ can be determined by substituting $x = x_{k-1}^*$. Clearly

$$T_{k-1}(t, x_{k-1}^*) = tT_k(x_{k-1}^*) + I_{k-1}(x_{k-1}^*) = tT_k(x_{k-1}^*),$$

implying the recursion

$$\text{(13)} \qquad t\varkappa_k(t) = \varkappa_{k-1}(t) - \frac{1-t}{t} E\left(\iota^{t-1}\mathbf{I}(\iota > x_{k-1}^*)\right) + \frac{x_{k-1}^{*t}}{t} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x_{k-1}^*)\right).$$

The value of $\varkappa_1(t)$ is obtained by calculating $T_1(t, x)$ directly. From (12), it follows that

$$I_l(x) = E\left\{\frac{1}{\iota}\left[1 - \frac{1}{l!}\left(\ln\frac{\iota}{x}\right)^l\right]\mathbf{I}(\iota > x)\right\}$$

for $x_1^* \leqq x < x_0^*$, thus

$$T_1(t, x) = \frac{1}{t(1-t)} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x)\right) - \frac{x^{-t}}{t} E\left(\iota^{t-1}\mathbf{I}(\iota > x)\right),$$

hence $\varkappa_1(t) = 0$. Now (13) gives

$$\varkappa_k(t) = \sum_{j=1}^{k-1} t^{j-k-1} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x_j^*)\left(x_j^{*t} - (1-t)\iota^t\right)\right)$$

and

$$\text{(14)} \qquad \begin{aligned} x^t T_k(t, x) &= t^{-k}\sum_{j=1}^{k-1} t^{j-1} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x_j^*)\left(x_j^{*t} - (1-t)\iota^t\right)\right) + \\ &+ \frac{1}{t(1-t)} E\left(\frac{1}{\iota}\mathbf{I}(\iota > x)\left(x^t - (1-t)\iota^t\right)\right). \end{aligned}$$

In (14), the coefficients of negative powers of t must vanish. Since k is arbitrary, (4) follows immediately, expanding it in a power series in t, we arrive at (3). □

Proof of Theorem 4. *a)* Since $S_l(k) = TS_l(k)$ holds for $k < k_l^*$, we also have $(k-1)S_l(k-1) = (k-1)TS_l(k-1) = S_l(k)+(k-1)TS_l(k) = kS_l(k)$ for $k < k_l^*$. Thus

$$P_r^*(\lambda) = s_r(1) = S_r(1) = (k_r^*-1)S_r(k_r^*-1) = (k_r^*-1)TS_r(k_r^*-1) =$$
$$= (k_r^*-1)TG_r(k_r^*-1) = (k_r^*-1)\big(G_r(k_r^*-1)-c_{r,k_r^*-1}\big).$$

From (7) it follows that

$$(k_r^*-1)G_r(k_r^*-1) = (k_r^*-1)G_1(k_r^*-1)+(k_r^*-1)TS_{r-1}(k_r^*-1) =$$
$$= (k_r^*-1)G_1(k_r^*-1)+P_{r-1}^*(\lambda),$$

hence

$$P_r^*(\lambda) = \sum_{j=1}^{r} (k_j^*-1)\big(G_1(k_j^*-1)-c_{j,k_j^*-1}\big). \tag{15}$$

Returning to (8), one can see that

$$G_1(k) = \sum_{i=k}^{\infty} \frac{p_i}{i} = E\left(\frac{1}{\iota_\lambda}\mathbf{I}(\iota_\lambda \geqq k)\right).$$

Let $\lambda \to \infty$ in (15). Since $\lim\limits_{\lambda\to\infty} c_{j,k_j^*-1} = I_j(x_j^*) = 0$, assertion *a)* follows immediately.

b) Rewrite (4) in the form

$$\sum_{j=1}^{\infty} t^{j-1}E\left(\frac{1}{\iota}\,\mathbf{I}(\iota > x_j^*)\right)x_j^{*\,t} = \sum_{j=1}^{\infty} t^{j-1}E\big(\iota^{t-1}\mathbf{I}(x_j^* < \iota \leqq x_{j-1}^*)\big),$$

and let t tend to 1 from below. Then

$$\sum_{j=1}^{\infty} E\left(\frac{x_j}{\iota}\,\mathbf{I}(\iota > x_j^*)\right) = \sum_{j=1}^{\infty} P(x_j^* < \iota \leqq x_{j-1}^*) = P(\iota > 0).$$

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