SOME REMARKS ON THE CONSTRUCTION OF MINIMAL DIMENSIONAL CONTROL AND OBSERVATION MATRICES

by

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Introduction

In this paper linear control-observation systems will be examined. Consider the following system

$$(1)\qquad \dot{x} = Ax+Bu$$
$$y = Cx,$$

where

$$A:\ [a, b]\to R^{n\times n}$$
$$B:\ [a, b]\to R^{n\times m}$$
$$C:\ [a, b]\to R^{r\times n}$$

are time dependent matrices. Vectors x, u and y are called state, input and output respectively. Assume that matrix A is known but matrices B and C have to be determined.

In this paper two problems will be investigated:

1. Find the minimal value of $m(A)$ and matrix B such that system (1) is completely controllable. Furthermore, find the input u such that $x(a) = \xi$, $x(b) = 0$, where ξ is a given constant vector.

2. Find the minimal value of $r(A)$ and matrix C such that system (1) is observable. Furthermore find $x(a)$.

In the time independent case the values of $m(A)$ and $r(A)$ can be any integer between 1 and n. In this case either matrix $[B, AB, \ldots, A^{n-1}B]$ or matrix

$$\begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{pmatrix}$$

must have full rank. For example, if $A = 0$, then obviously rank (B) or rank (C) should be equal to n_j and if A is a permutation matrix, then B and C can be selected as any basis vector e_i.

In the time dependent case, that is, in the case, when time dependent matrices B and C are permitted, we shall verify that always $m(A) = r(A) = 1$.

The Main Theorem

Theorem. *For any $n \times n$ matrix, A,*

$$m(A) = r(A) = 1$$

and for any $m \geq 1$ and $r \geq 1$ there exist usually time dependent matrices B and C of the type $n \times m$ and $r \times n$ such that system (1) *is controllable and observable.* □

Proof. The controllability part of the theorem will be proven, the observability of the system can be discussed in an analogous manner. It is well known that system (1) is completely controllable if and only the Kalman matrix

$$W[a, b] = \int_a^b \Phi(b, t)B(t)B^*(t)\Phi^*(b, t)dt \tag{2}$$

is invertable, where Φ is the solution of the initial value problem

$$\dot{\Phi} = A\Phi, \Phi(a) = I.$$

Let m be any positive integer, and assume that B is an $n \times m$ matrix. Observe that matrix W is the Gram matrix

$$\begin{pmatrix} (\varphi_1, \varphi_1) & (\varphi_1, \varphi_2) \cdots & (\varphi_1, \varphi_n) \\ \vdots & & \\ (\varphi_n, \varphi_1) & (\varphi_n, \varphi_2) \cdots & (\varphi_n, \varphi_n) \end{pmatrix} \tag{3}$$

associated with rows of $\Phi(b, t)\, B(t)$ according to the usual inner product of the space $L^2([a, b], R^m)$. It is well known that the Gram matrix is invertable if and only if functions $\varphi_1, \ldots \varphi_n$ are linearly independent, that is,

$$\alpha_1\varphi_1 + \ldots + \alpha_n\varphi_n = \mathbf{0}$$

can be true for $\alpha_i = 0$ $(1 \leq i \leq n)$, where $\mathbf{0}$ denotes the zero function. Hence matrix W is invertable if and only if

$$B(t) = \Phi^{-1}(b, t)\begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix}, \tag{4}$$

where functions $\varphi_1, \ldots, \varphi_n$ are linearly independent. For example, if the range of functions φ_i is in the real line, then $k = 1$, and equation (4) presents a matrix B such that system (1) is completely controllable. □

Corrollary. *If $x(a) = \xi$ is given then by selecting the input*

$$u(t) = -B^*(t)\,\Phi(b, t)^*\,\Phi(b, a)\xi \tag{5}$$

we shall have the required condition $x(b) = 0$. □

Proof. Using the Cauchy formula we get

$$x(b) = \Phi(b, a)\xi + \int_a^b \Phi(b, t)B(t)[-B^*(t)\Phi^*(b, t)\Phi(b, a)\xi]dt =$$

$$= \Phi(b, a)\xi - W[a, b]\Phi(b, a)\xi = 0. \quad \square$$

Remark 1. In the case of observability matrix C should be selected as

$$C(t) = (\Phi(t, a)^*)^{-1}\begin{pmatrix} \psi_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \psi_n(t) \end{pmatrix} \tag{6}$$

where functions $\psi_1, \ldots, \psi_n$ are linearly independent. In this case

$$\xi = \int_a^b \Phi(t, a)^*C^*(t)\left[y(t) - C(t)\int_a^b \Phi(t, \tau)B(\tau)u(\tau)d\tau\right]dt \tag{7}$$

gives the initial state of the system.

Remark 2. In applying equation (6) there is no need for the numerical inversion of matrix $\Phi(t, a)^*$, since the solution of the matrix initial value problem

$$\dot{Y} = -YA, \quad Y(a) = 1$$

equals the inverse of $\Phi(t, a)$, which is a simple consequence of the theory of adjoint equations.

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