

APPROXIMATE SOLUTION OF THE INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

By

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1. Assumptions and procedures of the method

Consider the initial value problem

$$(1) \quad y' = f(x, y), \quad 0 = x_0 \leq x \leq 1 \quad \text{and} \quad y(0) = y_0.$$

We assume that $f: (x_0 - \alpha, x_0 + \alpha) \times \mathbf{R} \rightarrow \mathbf{R}$ is defined and continuous with its first r derivatives, where $\alpha > 1$.

For $q = 0, 1, \dots, r$ we define $f^{[q]}$ by the algorithm:

$$f^{[0]} = f,$$
$$f^{[q]} = \frac{\partial}{\partial x} f^{[q-1]} + f \frac{\partial}{\partial y} f^{[q-1]}.$$

We remark that the derivatives of y , the solution of (1), can be expressed by the help of $f^{[q]}$ as follows:

$$y^{(q)}(x) = f^{[q-1]}(x, y(x)).$$

We also assume for $|x - x_0| < \alpha$ and $y, y_1, y_2 \in \mathbf{R}$

$$(2) \quad |f^{[q]}(x, y)| \leq M, \quad q = 0, 1, \dots, r$$

and the Lipschitz condition

$$(3) \quad |f^{[q]}(x, y_1) - f^{[q]}(x, y_2)| \leq L|y_1 - y_2|, \quad q = 0, 1, \dots, r$$

where L and M are some constants.

Consider the interval $0 \leq x \leq 1$ and define the mesh Δ by

$$\Delta: 0 = x_0 < x_1 < x_2 < \dots < x_k < x_{k+1} < \dots < x_n = 1$$

and let

$$x_{k+1} - x_k = h, \quad k = 0, 1, \dots, n-1.$$

The modulus of continuity of $y^{(r+1)}$ is denoted by ω_r .

Choosing the required positive integer m (the order of the error to be achieved), we define the function approximating the solution of (1) as $S_A(x)$ given by:

$$(4) \quad S_A(x) = S_k^{(m)}(x) = S_{k-1}^{(m)}(x_k) + \int_{x_k}^x f(t, S_k^{(m-1)}(t)) dt$$

where $x_k \leq x < x_{k+1}$, $k = 0, 1, \dots, n-1$ and $S_{-1}^{(m)}(x_0) = y_0$.

The following m iteration processes are considered in the above formula (4) for $x_k \leq x < x_{k+1}$ and $k = 0, 1, \dots, n-1$:

$$(5) \quad S_k^{(0)}(x) = S_{k-1}^{(m)}(x_k) + \sum_{q=0}^r \frac{f^{[q]}(x_k, S_{k-1}^{(m)}(x_k))}{(q+1)!} (x-x_k)^{q+1},$$

$$(6) \quad S_k^{(j)}(x) = S_{k-1}^{(m)}(x_k) + \int_{x_k}^x f(t, S_k^{(j-1)}(t)) dt, \quad j = 1, 2, \dots, m.$$

2. Error estimations and convergence

The exact solution of (1) can be written in the following forms, for $x_k \leq x < x_{k+1}$ and $k = 0, 1, \dots, n-1$:

$$(7) \quad y(x) = y(x_k) + \sum_{q=0}^{r-1} \frac{f^{[q]}(x_k, y_k)}{(q+1)!} (x-x_k)^{q+1} + \frac{f^{[r]}(\xi_k, y(\xi_k))}{(r+1)!} (x-x_k)^{r+1},$$

where $y_k = y(x_k)$ and $x_k < \xi_k < x_{k+1}$,

$$(8) \quad y(x) = y_k + \int_{x_k}^x f[t, y(t)] dt.$$

If we denote for $x_k \leq x < x_{k+1}$

$$|S_k^{(m)}(x) - y(x)| = e(x) \quad \text{and} \quad |S_k^{(m)}(x_k) - y(x_k)| = e_k,$$

then (6) and (8) together with the Lipschitz condition give

$$\begin{aligned} e(x) &\leq e_k + L \int_{x_k}^x |S_k^{(m-1)}(t_1) - y(t_1)| dt_1 \leq e_k + L \int_{x_k}^x \{|S_{k-1}^{(m)}(x_k) - y_k| + \\ &\quad + L \int_{x_k}^{t_1} |f[t_2, S_k^{(m-2)}(t_2)] - f[t_2, y(t_2)]| dt_2\} dt_1 \leq \dots \leq \\ &\leq e_k \sum_{j=0}^{m-1} \frac{L^j h^j}{j!} + L^m \int_{x_k}^x \int_{x_k}^{t_1} \dots \int_{x_k}^{t_{m-1}} |S_k^{(0)}(t_m) - y(t_m)| dt_m \dots dt_1. \end{aligned}$$

Using (5) and (7) and $y^{(q)}(x) = f^{[q-1]}(x, y(x))$, it is easy to prove that

$$|S_k^{(0)}(t_m) - y(t_m)| \leq e_k + Le_k \sum_{q=0}^{r-1} \frac{(t_m - x_k)^{q+1}}{(q+1)!} + \frac{\omega_r(h)}{(r-1)!} (t_m - x_k)^{r+1}.$$

Thus, for $c_0 = Le^L$ if $L > 1$ and $c_0 = e$ if $L \leq 1$, it is easy to get the following inequalities for $x_k \leq x \leq x_{k+1}$:

$$e(x) = |S_k^{(m)}(x) - y(x)| \leq e_k(1 + c_0h) + \frac{L^m}{(m+r+1)!} \omega_r(h)h^{m+r+1},$$

and consequently

$$e_k(1 + c_0h) \leq e_{k-1}(1 + c_0h)^2 + \frac{L^m}{(m+r+1)!} \omega_r(h)h^{m+r+1}(1 + c_0h),$$

$$e_1(1 + c_0h)^k \leq e_0(1 + c_0h)^{k+1} + \frac{L^m}{(m+r+1)!} \omega_r(h)h^{m+r+1}(1 + c_0h)^k$$

and $c_0 = 0$ gives

$$(9) \quad |S_k(x) - y(x)| \leq \frac{L^m}{(m+r+1)!} \omega_r(h)h^{m+r+1} \sum_{j=0}^k (1 + c_0h)^j$$

and thus we have proved the following theorem.

Theorem 1. *Let $y(x)$ be the exact solution of (1). If $S_A(x)$ given in (4) is the approximate solution, then the inequality*

$$|S_A(x) - y(x)| \leq c_1 \omega_1(h) h^{m+r}$$

holds for all $x \in [0, 1]$, where $c_1 = \frac{L^m(e^{c_0} - 1)}{c_0(m+r+1)!}$ and $c_0 = Le^L$ if $L < 1$, and $c_0 = e$, if $L \leq 1$. \square

Theorem 2. *The inequalities*

$$\left| \frac{d^{q+1}}{dx^{q+1}} S_A(x) - \frac{d^{q+1}}{dx^{q+1}} y(x) \right| \leq c_2 \omega_r(h) h^{m+r+1} \text{ if } m > 1,$$

$$\left| \frac{d^{q+1}}{dx^{q+1}} S_A(x) - \frac{d^{q+1}}{dx^{q+1}} y(x) \right| \leq c_2 \omega_r(h) h^{r+1} \text{ if } m = 1$$

hold for all $x \in [0, 1]$ and all $q = 0, 1, \dots, r$, where $c_2 = (m+r+1)c_1$. \square

Proof. From (4) and (8), using the Lipschitz condition we get

$$\left| \frac{d^{q+1}}{dx^{q+1}} S_A(x) - \frac{d^{q+1}}{dx^{q+1}} y \right| \leq L |S_k^{(m-1)}(t_1) - y^{(m-1)}(t_1)|$$

and applying theorem 1, taking into consideration $m-1$ instead of m , it is easy to complete the proof. \square

Theorem 3. *The error by which $S_{\Delta}(x)$ fails to satisfy the differential equation (1) is given by the inequality*

$$|S'(x) - f(x, S_{\Delta}(x))| \leq c_3 \omega_r(h) h^{m+r-1}$$

which holds for all $x \in [0, 1]$ where $c_3 = Lc_1 + c_2$. \square

Proof. Adding and subtracting $y'(x)$ and using Theorems 1 and 2, it is easy to prove Theorem 3. \square

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