ON THE SMOOTHNESS PROPERTIES OF STATIONARY FUNCTIONS ARISING IN CALCULUS OF VARIATIONS

by

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It is known that in one-dimensional regular problems of calculus of variations the solutions of the Euler-Lagrange differential equation provide a relative extremum for the corresponding functional, at least "in small". Concerning this it is worth pointing out the following interesting fact. From the classical conditions of calculus of variations (see [1], [2] and [5]) it follows that if we take certain rather substantial extensions of a functional defined on twice continuously differentiable functions then the extensions take a relative extremum at the mentioned solutions of the Euler-Lagrange differential equations as well. Of course, this does not mean that a functional defined on a rather general set (consisting e.g. of piece-wise smooth or absolutely continuous functions) can attain an extremum only on "nice", namely twice continuously differentiable functions. There exist well-known examples for functionals attaining extrema on piece-wise smooth functions (see [5]). For this reason, it is important to determine the types of functionals which can attain their extrema only on the solutions of the Euler-Lagrange differential equation. This is important from the point of view of numerical mathematics, too, because in this case any function providing an extremum can be obtained via solving a differential equation.

In this field, the first essential result is due to Hilbert (see [3]). He proved that if we extend a one-dimensional regular functional defined on twice continuously differentiable functions in such a way that we "admit" once continuously differentiable functions as well, then this extension can not have any new extremal function. Hilbert's result can be generalized in several ways. In the present paper we give the following plausible generalization. The domain of the functional is defined as the largest class of functions where the admissible continuous curves are supposed to have both left and right hand tangents at each point.

For the present treatment we choose the simplest one-dimensional non-parametric functionals. However, the results can easily be extended to more general cases.

1. Notations and definitions

Let $a, b, c, d \in \mathbf{R}$, $a<b$. Denote by C_0 and C_i the classes of all continuous respectively i times continuously differentiable real functions on $[a, b]$ $(i = 1, 2)$. Let F be the class of all real functions of first kind defined on $[a, b[$, i.e., $\varphi \in F$ means that φ has a finite right limit at every point of the interval $[a, b]$ and a finite left limit at every point of the interval $]a, b]$. The elements of the set F are called also regulated functions (see [4]). Denote by F_1 the set consisting of the integral functions of functions in F. For a $\varphi \in F$ we shall use the following notation

$$\int_a \varphi : [a, b] \to \mathbf{R}, \quad t \mapsto \int_a^t \varphi.$$

Let $j := id_{[a,b]}$ and for every $\alpha \in \mathbf{R}$ define

$$\underset{\sim}{\alpha} : [a, b] \to \mathbf{R}, \quad t \mapsto \alpha.$$

Now introduce the following sets

$$M_0 := \{x \in F_1 | x(a) = c, \quad x(b) = d\},$$
$$M_i := \{x \in C_i | x(a) = c, \quad x(b) = d\} \quad (i = 1, 2);$$
$$H_0 := \{x \in F_1 | x(a) = x(b) = 0\},$$
$$H_i := \{x \in C_i | x(a) = x(b) = 0\} \quad (i = 1, 2).$$

For any function $x \in F_1$ define

$$\overset{1}{x} := (j, x, \dot{x}).$$

Let $f : \mathbf{R}^3 \to \mathbf{R}$ be twice continuously differentiable and use the following convention

$$f\overset{1}{x} := f \circ \overset{1}{x} \quad (x \in F_1).$$

Denote by $f_{\cdot i}$ the partial derivative function of f with respect to the i-th variable and by $f_{\cdot ik}$ the second partial derivative function of f with respect to the i-th and k-th variables $(i, k = 1, 2, 3)$.

Let φ and ψ be real functions the domains of which are equal to the interval $[a, b]$ except for an at most countable set. We shall write

$$\varphi \doteq \psi$$

if there exists an at most countable set $A \subset [a, b]$ such that

$$\varphi(t) = \psi(t) \quad (t \in [a, b] \setminus A).$$

Now for $i = 1, 2, 3$ define

$$\mathcal{J}_i : M_i \to \mathbf{R}, \quad x \mapsto \int_a^b f\overset{1}{x}.$$

Definition 1. The function f is said to be (positively) *regular* if for every $(t, s, u) \in \mathbf{R}^3$

$$f_{\cdot 33}(t, s, u) > 0.$$

Definition 2. Let $i = 0, 1, 2$ and $\psi \in M_i$ be fixed. The functional

$$\delta_\varphi \mathcal{J}_i : H_i \to \mathbf{R}, \quad h \mapsto \int_a^b \left[\left(f_{\cdot 2}\overset{1}{\varphi}\right) h + \left(f_{\cdot 3}\overset{1}{\varphi}\right) h' \right]$$

is called the *first variation* of $\mathcal{J}_i$ corresponding to φ.

Definition 3. If for an $i = 0, 1, 2$ and $\varphi \in M_i$ the range of $\delta_\varphi \mathcal{J}_i$ is the set $\{o\}$ then φ is called a *stationary function* of $\mathcal{J}_i$. The set of all stationary functions of the functional $\mathcal{J}_i$ is denoted by the symbol

$$\mathit{Stat}\, \mathcal{J}_i \quad (i = 0, 1, 2).$$

Definition 4. Define for any $\varphi, \psi \in F_1$

$$\varrho(\varphi, \psi) := \max_{t \in [a, b]} |\varphi(t) - \psi(t)| + \sup_{t \in [a, b[} |\varphi'(x+o) - \psi'(x+o)| + \sup_{t \in]a, b]} |\varphi'(x-o) - \psi'(x-o)|.$$

It is easy to see that ϱ is a metrics on F_1.

Definition 5. Fix $i = 0, 1, 2$. We shall say that $\mathcal{J}_i$ attains a relative weak minimum on the function $\varphi \in M_i$ if there exists a positive real number ε such that for every $\psi \in M_i$ with $\varrho(\varphi, \psi) < \varepsilon$ we have

$$\mathcal{J}_i(\psi) \geq \mathcal{J}_i(\varphi).$$

2. Results

Theorem 1 (*Du Bois — Reymond*). *Let $m \in F$ and suppose that for every $h \in H_0$*

$$\int_a^b mh' = 0. \tag{1}$$

Then there exists an $\alpha \in \mathbf{R}$ such that

$$m \doteq \underset{\sim}{\alpha}. \tag{2}$$

The theorem can be proved in the usual way (see [2], [5]): for an appropriate $\alpha \in \mathbf{R}$ the function

$$\bar{h} := \int_a (m - \underset{\sim}{\alpha})$$

belongs to H_0. Then by (1) we get

$$\int_a^b m\bar{h}' = \int_a^b (m - \underset{\sim}{\alpha})\bar{h}' = \int_a^b (m - \underset{\sim}{\alpha})^2 = 0.$$

Hence, taking into account that $(m-\underset{\sim}{\alpha})^2$ is a nonnegative function of first kind, it follows that m is equal to zero at any of its points of continuity (i.e. except for an at most countable set, see [4] and [7]). So (2) holds.

Fix the number $i = 0, 1, 2$ and suppose that $\mathcal{J}_i$ attains a relative weak minimum on some $\varphi \in M_i$. Then, as it is easy to see, the Gateaux derivative of the functional $\mathcal{J}_i$ is equal to zero in any direction $h \in H_i$ (see [5], [6]). It can be shown in the usual way that this derivative is $\delta_\varphi \mathcal{J}_i(h)$. So

$$\delta_\varphi \mathcal{J}_i(h) = 0 \quad (h \in H_i).$$

This result can also be formulated in the following way.

Theorem 2. *Fix $i = 0, 1, 2$ and suppose that $\mathcal{J}_i$ attains a relative weak minimum on $\varphi \in M_i$. Then*

$$\varphi \in \mathit{Stat}\ \mathcal{J}_i.$$

Theorem 3 *(Euler – Lagrange). $\varphi \in \mathit{Stat}\ \mathcal{J}_0$ if and only if there exists an $\underset{\sim}{\alpha} \in \mathbf{R}$ such that*

$$f \cdot {}_3\overset{1}{\varphi} \doteq \int_a f \cdot {}_2\overset{1}{\varphi} + \underset{\sim}{\alpha}. \tag{3}$$

Proof. Suppose that (3) holds. Then for any $h \in H_0$

$$\delta_\varphi \mathcal{J}_0(h) = \int_a^b \left[\left(f \cdot {}_2\overset{1}{\varphi}\right) h + \left(f \cdot {}_3\overset{1}{\varphi}\right) h' \right]. \tag{4}$$

If we partially integrate the first member then by (3) we obtain that

$$\delta_\varphi \mathcal{J}_0(h) = \int_a^b \left[-\int_a f \cdot {}_2\overset{1}{\varphi} + f \cdot {}_3\overset{1}{\varphi} \right] h' = 0,$$

that is $\varphi \in \mathit{Stat}\ \mathcal{J}_0$.

Conversely, suppose that $\varphi \in \mathit{Stat}\ \mathcal{J}_0$ i.e. the right-hand side of (4) is zero for any $h \in H_0$. Then, like above, by partial integration we get that

$$\int_a^b \left(f \cdot {}_3\overset{1}{\varphi} - \int_a f \cdot {}_2\overset{1}{\varphi} \right) h' = 0 \quad (h \in H_0).$$

Hence, by Theorem 1, is follows that (3) holds. □

We note that from classical results of calculus of variations (see [2], [5]) it follows that

a) $\varphi \in \mathit{Stat}\ \mathcal{J}_1$ if and only if

$$f \cdot {}_2\overset{1}{\varphi} - \left(f \cdot {}_3\overset{1}{\varphi}\right)' = \underset{\sim}{0}, \tag{5}$$

b) $\varphi \in Stat\, \mathcal{J}_2$ if and only if

$$f_{\cdot 2}\overset{1}{\varphi} - f_{\cdot 31}\overset{1}{\varphi} - f_{\cdot 32}\overset{1}{\varphi} \cdot \varphi' - f_{\cdot 33}\overset{1}{\varphi} \cdot \varphi'' = \underset{\sim}{0}. \tag{6}$$

Theorem 4 *(Weierstrass – Erdmann). Let* $\varphi \in Stat\, \mathcal{J}_0$. *Then for every* $t \in \,]a, b[$ *we have*

$$f_{\cdot 3}(t, \varphi(t), \varphi'(t-0)) = f_{\cdot 3}(t, \varphi(t), \varphi'(t+0)). \tag{7}$$

Proof. Since $\varphi \in Stat\, \mathcal{J}_0$ then relation (3) holds. The function

$$\int_a f_{\cdot 2}\overset{1}{\varphi} + \underset{\sim}{\alpha}$$

is continuous at each point of $[a, b]$ so by the definition of the relation $\doteq$ we get

$$\lim_{t-0} f_{\cdot 3}\overset{1}{\varphi} = \lim_{t+0} f_{\cdot 3}\overset{1}{\varphi} \quad (t \in \,]a, b[).$$

Since f is continuous we have

$$f_{\cdot 3}\Big(\lim_{t-0} \overset{1}{\varphi}\Big) = f_{\cdot 3}\Big(\lim_{t+0} \overset{1}{\varphi}\Big) \quad (t \in \,]a, b[).$$

If we rewrite the last equality in details then we obtain just the "corner condition" (7). □

Theorem 5. *Let* $\varphi \in Stat\, \mathcal{J}_0$ *and suppose that for some* $t \in \,]a, b[$ *the function*

$$\mathbf{R} \ni u \mapsto f_{\cdot 3}(t, \varphi(t), u) \in \mathbf{R} \tag{8}$$

is strictly monotone. Then φ *is continuously differentiable at t.*

Proof. Since $\varphi \in F_i$ then φ' has both left and right limit at t. Hence φ is continuously differentiable at t if and only if

$$\varphi'(t-0) = \varphi'(t+0).$$

Suppose the contrary, $\varphi'(t-0) \neq \varphi'(t+0)$. Then the fact that the function (8) is strictly monotone implies that

$$f_{\cdot 3}(t, \varphi(t), \varphi'(t-0)) \neq f_{\cdot 3}(t, \varphi(t), \varphi'(t-0))$$

in contradiction to equality (7). □

Theorem 6 (*Hilbert*). *If f is regular then*

$$Stat\, \mathcal{J}_1 = Stat\, \mathcal{J}_2.$$

For the proof see e.g. [5].

After the above preparation we can announce the following result.

Theorem 7. *If f is regular then*

$$Stat\,\mathcal{J}_0 = Stat\,\mathcal{J}_2.$$

Proof. By Hilbert's theorem it is enough to verify that

$$Stat\,\mathcal{J}_0 = Stat\,\mathcal{J}_1.$$

Here clearly $Stat\,\mathcal{J}_1 \subset Stat\,\mathcal{J}_0$, so it is enough to see that

$$Stat\,\mathcal{J}_0 \subset Stat\,\mathcal{J}_1.$$

Let $\varphi \in Stat\,\mathcal{J}_0$. Since $f._{33}$ is a positive function then for any $t \in]a, b[$ the corresponding function (8) is strictly monotone increasing. Hence, by Theorem 5, it follows that φ is continuously differentiable at any point of the interval $]a, b[$.

A simple calculation shows that φ is continuously differentiable at a as well i.e. we have

$$\varphi'(a) = \lim_a \varphi'. \tag{9}$$

Suppose the contrary, let e.g.

$$\varphi'(a) < \lim_a \varphi'. \tag{10}$$

Since φ is differentiable at a then there exists a function $\Phi: [a, b] \to \mathbf{R}$, continuous at a such that

$$\varphi(x) - \varphi(a) = \Phi(x)(x-a) \quad (x \in [a, b]) \tag{11}$$

where $\varphi'(a) = \Phi(a)$. On the other hand

$$\varphi(x) - \varphi(a) = \int_a^x \varphi' \quad (x \in [a, b]). \tag{12}$$

From the inequality (10) it follows that there exist a real number A and a positive number δ such that on the one hand by (11) we have

$$\varphi(x) - \varphi(a) < A(x-a) \quad (x \in]a, a+\delta]). \tag{13}$$

On the other hand, by (12)

$$\varphi(x) - \varphi(a) > A(x-a) \quad (x \in]a, a+\delta]). \tag{14}$$

(13) and (14) clearly contradict each other, so (9) holds. In quite a similar way it can be proved that φ is continuously differentiable at the point b as well.

Thus, we have shown that $\varphi \in C_1$. This implies that $f._3\overset{1}{\varphi}$ is a continuous function defined on $[a, b]$. The definition of the relation $\doteq$ implies that in (3) the role of the relation $\doteq$ turns into the "ordinary" relation $=$,

$$f._3\overset{1}{\varphi} = \int_a f._2\overset{1}{\varphi} + \underset{\sim}{\alpha}.$$

By the continuity of $f\cdot_2\overset{1}{\varphi}$, the right hand side is differentiable, thus so is the left-hand side,

$$f\cdot_2\overset{1}{\varphi}-\left(f\cdot_3\overset{1}{\varphi}\right)'=\underset{\sim}{0}.$$

The last equality just means (see [5]) that $\varphi\in Stat\,\mathcal{J}_1$. □

Remark. According to Theorem 2 in case of a regular f the functional $\mathcal{J}_0$ can reach an extremum only on a twice continuously differentiable function, i.e. the Euler – Lagrange differential function (G) must be satisfied.

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