THE APPROXIMATE SOLUTION OF NON-LINEAR FUNCTIONAL EQUATIONS BY A STEFFENSEN-TYPE METHOD

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The classic secant method can be extended for solving a non-linear operator equation defined in the conditions of Banach spaces, by using the generalized divided differences treated in the last chapter of the monography [3]. In their paper [2] M. Balázs and G. Goldner proposed the

$$x_{n+1} = x_n - \frac{F(x_n)}{F_{x_n x_{n-1}}(y_n)} y_n \quad (n = 0, 1, \dots) \tag{1}$$

method for the computation of the solution of the equation

$$F(x) = 0, \tag{2}$$

where $F: X \to \mathbf{R}$, (X being a Banach space) is a non-linear functional, and the auxiliary vector series was chosen in the following form

$$y_n := \frac{x_n - x_{n-1}}{\|x_n - x_{n-1}\|}, \quad (n = 0, 1, \dots). \tag{3}$$

In this paper we are going to solve the functional equation (2) using a Steffensen-type method instead of the above secant method. Let's define a non-linear operator $\Phi : X \to X$ in the X Banach space using the above mentioned functional F

$$\Phi(x) := x - F(x) \frac{x}{\|x\|} \tag{4}$$

and let x_n, u_n be the nodal points of our method, where x_n is the n-th approximate solution of (2), but u_n is computed by the following form

$$u_n := \Phi(x_n). \tag{5}$$

We define step by step a scalar L_n and a vector y_n in the following forms

$$
\left.\begin{aligned}
L_n &:= \frac{1}{F_{x_n u_n}(y_n)}, \\
y_n &:= \frac{x_n - u_n}{\|x_n - u_n\|},
\end{aligned}\right\} \quad (n = 0, 1, \ldots) \tag{6}
$$

and suggest the following method

$$x_{n+1} = x_n - L_n F(x_n) y_n \quad (n = 0, 1, \ldots) \tag{7}$$

for solving the equation (2). According to the essential feature of divided differences $F_{x'x''}(x'-x'') = F(x') - F(x'')$ the following relation

$$\|F_{x_n u_n}\| = |F_{x_n u_n}(y_n)|, \quad (n = 0, 1, \ldots), \tag{8}$$

can be proved easily. Further on we assume the functional F as being continuous in the sphere $S(x_0, r)$ specified later on.

Theorem. *We suppose that the following conditions are satisfied*

1°. *For the divided difference of the functional F in the points x_0, u_0 we have*

$$|L_0| = \frac{1}{\|F_{x_0 u_0}\|} \le B_0, \quad B_0 \ge 1\,.$$

2°. *For the approximation x_0*

$$|F(x_0)| \le \eta_0\,.$$

3°. *The divided difference of the operator Φ and the divided difference of the second order of the functional F satisfy the following relations*

$$\|\Phi_{x'x''}\| \le M; \quad \|F_{x'x''} - F_{x''x'''}\| \le K\,\|x' - x'''\|$$

if $x', x'', x''' \in S(x_0, r)$, where

$$r := \max\left\{\frac{B_0\,\eta_0(1-\alpha)}{1-2\alpha};\ \frac{MB_0\,\eta_0(1-\alpha)}{1-2\alpha} + \eta_0\right\}.$$

4°. *The constant numbers B_0, η_0, K, M satisfy the inequality*

$$h_0 := 2K\,B_0^2\,\eta_0(M+1) < \alpha < \frac{3-\sqrt{5}}{2}\,.$$

Then equation (2) has at least a solution $x^* \in S(x_0, r)$ which is the limit of the approximations (7) and the rapidity of the convergence is characterized by the inequality

$$\|x_n - x^*\| \le (1-\alpha)^n \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2^n - 1} \cdot \delta\,, \tag{9}$$

where

$$\delta := B_0 \eta_0 \sum_{k=1}^{\infty} (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2} \right)^{2(2^{k-1}-1)} .$$

Proof. First of all let's take the following estimates, which are given directly from the formulas (5), (7) and the assumptions of Theorem:

$$\begin{aligned} \|x_1 - x_0\| &= \|L_0 F(x_0) y_0\| \le B_0 \eta_0 , \\ \|u_1 - u_0\| &= \|\Phi(x_1) - \Phi(x_0)\| \le M B_0 \eta_0 , \\ \|x_1 - u_0\| &\le \|x_0 - u_0\| + \|L_0 F(x_0) y_0\| \le \eta_0 + B_0 \eta_0 \le 2 B_0 \eta_0 , \\ \|u_1 - x_0\| &\le M B_0 \eta_0 + \eta_0 . \end{aligned} \tag{10}$$

According to formula (7) it results immediately

$$F_{x_0 u_0} (x_1 - x_0) = - F_{x_0 u_0} (L_0 F(x_0) y_0) ,$$

whence using the formulas (6) and the definition of the divided differences we obtain the identity

$$F_{x_0 u_0} (x_1 - x_0) = - F(x_0) . \tag{11}$$

Substituting x_0 for x_1 from the conditions of this theorem, we have

$$\|F_{x_1 u_1}\| \ge \|F_{x_0 u_0}\| \left(1 - \frac{K (B_0 \eta_0 + M B_0 \eta_0)}{\|F_{x_0 u_0}\|} \right) \ge \frac{1 - h_0}{B_0} ,$$

whence it results

$$|L_1| = \frac{1}{\|F_{x_1 u_1}\|} \le \frac{B_0}{1 - h_0} =: B_1 . \tag{12}$$

From (11) we obtain

$$|F(x_1)| \le K \|x_1 - u_0\| \, \|x_1 - x_0\| \le 2 K B_0^2 \eta_0^2 (M + 1) ,$$

so we have

$$h_0 \eta_0 =: \eta_1 . \tag{13}$$

Considering the formulas (12), (13)

$$h_1 := 2 K B_1^2 \eta_1 (M + 1) \le \frac{h_0^2}{(1 - h_0)^2} \tag{14}$$

is given. The inequality $h_1 < \alpha$ is satisfied iff $\alpha < \frac{3 - \sqrt{5}}{2}$. By induction we obtain

$$|L_n| \leq B_n \leq \frac{B_{n-1}}{1-\alpha} \leq \frac{1}{(1-\alpha)^n} B_0,$$

(15)
$$|F(x_n)| \leq \eta_n \leq h_{n-1}\,\eta_{n-1} \leq \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2^n-1} (1-\alpha)^{2n}\,\eta_0,$$

$$h_n \leq \frac{h_{n-1}^2}{(1-h_{n-1})^2} \leq \left(\frac{1}{1-\alpha}\right)^{2(2^n-1)} h_0^{2^n}.$$

On the basis of the formulas (15) we have

$$\|x_{n+p}-x_n\| \leq B_{n+p-1}\,\eta_{n+p-1} + \dots + B_n\,\eta_n \leq$$
$$\leq B_0\,\eta_0 \sum_{k=1}^{p} \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2^{n+k-1}-1} (1-\alpha)^{n+k-1},$$

whence using the inequality

$$(2^{n+k-1}-1)-(2^n-1) \geq 2(2^{n-1}-1), \quad n \geq 1,$$

we obtain

(16)
$$\|x_{n+p}-x_n\| \leq (1-\alpha)^n \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2^n-1} \cdot \delta_p,$$
$$\delta_p := B_0\,\eta_0 \sum_{k=1}^{p} (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2(2^{k-1}-1)}.$$

Considering the completeness of X it results that the sequence $\{x_n\}$ has a limit $x^* \in X$. Since $\lim_{n\to\infty} \eta_n = 0$ by the continuity of the functional F we obtain $F(x^*)=0$. From (16) if $p\to\infty$ the rapidity of convergence is given.

Using the inequality $2^{k-1} \geq k$, $k \geq 1$ we can estimate $\|x_n-x_0\|$ and $\|u_n-x_0\|$ in the same way as (16). So we have

(17)
$$\|x_n-x_0\| \leq B_0\,\eta_0 \sum_{k=1}^{n} (1-\alpha)^{k-1} \left(\frac{\alpha}{(1-\alpha)^2}\right)^{2^{k-1}-1} \leq \frac{1-\alpha}{1-2\alpha} B_0\,\eta_0,$$
$$\|u_n-x_0\| \leq \frac{MB_0\,\eta_0(1-\alpha)}{1-2\alpha} + \eta_0.$$

Choosing r as in the 3°. condition of the Theorem it results $x^* \in S(x_0, r)$. This completes the proof.

Computational example

$$F(x) = (x^2+y^2-2)^2+(x^2-y^2)^2 = 0,$$

initial values: $\left.\begin{array}{l} x_{-1}\,(.7;\ .7) \\ x_0\ \ (.8;\ .8) \end{array}\right\}$ for the secant method,

x_0 (.7; .7) for the Steffensen-method.

	Secant method	Steffensen method
1	.89931 .89931	.959508 .959508
2	.938356 .938356	.976706 .976706
3	.963359 .963359	.987196 .987196
4	.977575 .977575	.993213 .993213
5	.98629 .98629	.99649 .99649
6	.991566 .991566	.998227 .998227
7	.994807 .994807	.99913 .99913

The exact solution is (1; 1), the secant method will accurate to three decimal places after 11 step.

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