

PARALLEL PROCESSING OF 0—1 SEQUENCES

By

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§. 1. Introduction

This paper is devoted to the investigation of parallel processing of binary sequences.

Processing proceeds in the points of time 1, 2, . . . , and it is sequential for every sequence.

The sequences have priorities: processing algorithm considers the elements when choosing them for processing at every point of time in the order first, second. . . . , last sequence.

The processing in every point of time is characterized by using the vector of the first non-processed elements of the sequences.

In §. 2. the formal description of the considered problem is given, later (§. 3.) the limit distribution of the vectors of the first non-processed elements is computed.

Finally (§. 4.) the limit distribution is used to get the processing speed.

§. 2. Formulation of the problem

Let

$$(2.1) \quad \begin{array}{l} F_1 = f_{11}, f_{12}, \dots \\ \vdots \\ F_r = f_{r1}, f_{r2}, \dots \end{array}$$

r ($r \geq 2$) infinite binary sequences that is $f_{ij} \in \{0, 1\}$ ($i = 1, \dots, r; j = 1, 2, \dots$).

We process these sequences using the following algorithm [1].

1. Processing proceeds in the discrete points of time 1, 2, Let $t = 1$.
2. Let $B_t = (f_{1t}, f_{2t}, \dots, f_{rt})$.
3. In the moment of time t let us distinguish three cases:

- a) $f_{12} \neq f_{11}$;
 b) there exists an index k ($1 \leq k < r$) for which $f_{12} = f_{11} = f_{21} = \dots = f_{k1}$ and $f_{k+1,1} \neq f_{11}$;
 c) $f_{12} = f_{11} = f_{21} = \dots = f_{r1}$.

In the case a) the elements f_{11} and f_{12} , in the case b) the elements f_{11} and $f_{k+1,1}$, and in the case c) only the element f_{11} will be processed.

4. We omit the processed elements, reduce the second index of the remaining elements in the i -th sequence by the number of the omitted ones in the t -th point of time elements of the i -th ($i = 1, \dots, r$) sequence.

5. We add 1 to t and continue the processing from the Step 2.

Of course, the element f_{i1} of B_t ($t = 1, 2, \dots$) is the starting (first non-processed) element of F_i in the t -th point of time; we call B_1, B_2, \dots the state sequence of processing.

Let now

$$(2.2) \quad \begin{aligned} \Theta_1 &= \xi_{11}, \xi_{12}, \dots \\ &\vdots \\ \Theta_r &= \xi_{r1}, \xi_{r2}, \dots \end{aligned}$$

be r sequences of independent random variables with common distributions

$$(2.3) \quad P(\xi_{ij} = 0) = P(\xi_{ij} = 1) = \frac{1}{2} \quad (i = 1, \dots, r; j = 1, 2, \dots).$$

Let us suppose that the realisations of the sequences are processed using the algorithm given above.

If the vectors σ_t ($t = 1, 2, \dots$) are determined by the distribution of the possible values of B_t 's, then σ_t 's are random variables.

Our algorithm and scheme is a special case of those studied in [2]; in that work ergodicity was proved for the studied systems.

In [3] the variables ξ_{ij} are independent, uniformly distributed on the set $\{1, \dots, N\}$, and the question was considered: how many rows are needed to process N (different) elements? Limit distribution was given for this random variable for $t = 1, N \rightarrow \infty$. In [5] limit distribution is given for it if $t > 1$ fixed and $N \rightarrow \infty$.

For any such system the problem of the speed, especially the asymptotic speed (when $t \rightarrow \infty$) of the algorithm is of basic importance, as Katai pointed out [4]. The asymptotic speed depends on the ergodic behaviour of σ_t . As we mentioned already, σ_t is an ergodic Markov-chain, and to derive its ergodic distribution is our next aim in the case $N = 2, p = \frac{1}{2}$.

First of all let us compute the transition probabilities for this sequence in the case $r = 3$.

Then there are $2^3 = 8$ possible vectors: $(0, 0, 0), (0, 0, 1), \dots, (1, 1, 1)$. For the sake of simplicity we denote the vector (x_1, x_2, x_3) by

$$(2.4) \quad V = 4x_1 + 2x_2 + x_3, \quad V = 0, 1, \dots, 7 \quad (x_i = 0, 1).$$

The transition probabilities (only the positive ones) are presented in the following table.

Table 1.

Transition probabilities for $r = 3$								
	0	1	2	3	4	5	6	7
0	3/4	–	–	–	1/4	–	–	–
1	1/4	2/4	–	–	–	1/4	–	–
2	1/4	–	2/4	–	–	–	1/4	–
3	–	1/4	–	2/4	–	–	–	1/4
4	1/4	–	–	–	2/4	–	1/4	–
5	–	1/4	–	–	2/4	–	1/4	–
6	–	–	1/4	–	–	–	2/4	1/4
7	–	–	1/4	–	–	–	–	3/4

We get these probabilities as follows. For the vectors 0 and 7 we have $f_{11} = f_{21} = \dots = f_{r1}$, therefore only the cases *a*) and *c*) are possible in Step 3. of the algorithm. With probability $P(f_{11} \neq f_{12} \text{ and } f_{12} = f_{13}) = \frac{1}{4}$ we get the transitions $0 \rightarrow 4$ or $7 \rightarrow 3$ and with

$$(2.5) \quad P(f_{11} = f_{12}) + P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{11}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

we have the transitions $0 \rightarrow 0$ or $7 \rightarrow 7$, that is

$$(2.6) \quad P(0 \rightarrow 4) = P(7 \rightarrow 3) = \frac{1}{4} \quad \text{and} \quad P(0 \rightarrow 0) = P(7 \rightarrow 7) = \frac{3}{4}.$$

For the remaining vectors only the cases *a*) and *b*) are possible. Let

$$(2.7) \quad \bar{f}_{ij} = \begin{cases} 1, & \text{if } f_{ij} = 0, \\ 0, & \text{if } f_{ij} = 1, \end{cases}$$

and

$$(2.8) \quad \overline{f_1, f_2, \dots, f_r} = \bar{f}_1, \bar{f}_2, \dots, \bar{f}_r.$$

Then for the vector s ($1 \leq s \leq 6$) let $f_{11} = f_{21} = \dots = f_{k1}$ and $f_{k+1,1} \neq f_{11}$. Now

$$(2.9) \quad P(s \rightarrow s) = P(f_{11} = f_{12} \text{ and } f_{k+1,1} = f_{k+1,2}) + P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{11}) = \frac{2}{4},$$

$$(2.10) \quad \begin{aligned} P((f_{11}, \dots, f_{r1}) \rightarrow (f_{11}, f_{21}, \dots, f_{r1})) &= \\ &= P(f_{11} \neq f_{12} \text{ and } f_{13} = f_{12}) = \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} P((f_{11}, \dots, f_{k,1}, f_{k+1,1}, \dots, f_{r1}) \rightarrow (f_{11}, \dots, f_{k,1}, f_{k+1,1}, \dots, f_{r1})) &= \\ &= P(f_{11} = f_{12} \text{ and } f_{k+1,1} \neq f_{k+1,2}) = \frac{1}{4}. \end{aligned}$$

It is easy to see that the sequence $\sigma_1, \sigma_2, \dots$ is a homogeneous ergodic Markov-chain.

If

$$\sigma_t = (f_{11}, \dots, f_{r1}),$$

and

$$(2.11) \quad \begin{aligned} f_{11} = f_{12} = \dots = f_{1,r+1} &= 0, \\ f_{22} = f_{32} = \dots = f_{r2} &= 0, \end{aligned}$$

then $\sigma_{t+r} = (0, 0, \dots, 0)$, that is from any vector σ_t we can reach the vector $(0, 0, \dots, 0)$ in r transitions with a probability which is not less than $\left(\frac{1}{2}\right)^{2r}$, and this fact is enough to guarantee ergodicity due to the Markov-theorem.

In the following paragraph we shall determine the ergodic distribution, that is the limit probabilities

$$\lim_{t \rightarrow \infty} P(\sigma_t = j) = p_j \quad (j = 0, 1, \dots, 2^r - 1).$$

§. 3. The ergodic probabilities

Now we are going to determine the limit distribution of the vectors of the first non-processed elements.

If we process r sequences, then the vectors have 2^r possible values. The ergodic probabilities have to satisfy the following equations:

$$(3.1) \quad p_j = \sum_{i=1}^n p_i p_{ij} \quad (j = 0, 1, \dots, 2^r - 1)$$

and

$$(3.2) \quad \sum_{j=0}^{2^r-1} p_j = 1.$$

Before the general case let us consider the case $r = 3$, whose transition probabilities are given in Table 1.

In this case

$$(3.3) \quad p_0 = \frac{3}{4}p_0 + \frac{1}{4}p_1 + \frac{1}{4}p_2 + \frac{1}{4}p_4, \quad ,$$

$$(3.4) \quad p_1 = \frac{2}{4}p_1 + \frac{1}{4}p_3 + \frac{1}{4}p_5, \quad ,$$

$$(3.5) \quad p_2 = \frac{2}{4}p_2 + \frac{1}{4}p_6, \quad ,$$

$$(3.6) \quad p_3 = \frac{2}{4}p_3 + \frac{1}{4}p_7, \quad ,$$

$$(3.7) \quad p_4 = \frac{1}{4}p_0 + \frac{2}{4}p_4, \quad ,$$

$$(3.8) \quad p_5 = \frac{1}{4}p_1 + \frac{2}{4}p_5, \quad ,$$

$$(3.9) \quad p_6 = \frac{1}{4}p_2 + \frac{1}{4}p_4 + \frac{2}{4}p_6, \quad ,$$

$$(3.10) \quad p_7 = \frac{1}{4}p_3 + \frac{1}{4}p_5 + \frac{1}{4}p_6 + \frac{3}{4}p_7, \quad ,$$

and

$$(3.11) \quad p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1.$$

In consequence of the symmetry we have

$$(3.12) \quad p_7 = p_0, \quad p_6 = p_1, \quad p_5 = p_2 \quad \text{and} \quad p_4 = p_3,$$

and so instead of the system (3.3)–(3.11) we get

$$(3.13) \quad 4p_1 = 2p_1 + p_2 + p_3,$$

$$(3.14) \quad 4p_2 = p_1 + 2p_2,$$

$$(3.15) \quad 4p_3 = p_0 + 2p_3,$$

and

$$(3.16) \quad 2p_0 + 2p_1 + 2p_2 + 2p_3 = 1.$$

(We remark, that (3.3) was omitted because of the redundancy.)

From (3.14) and (3.15) we get

$$(3.17) \quad 2p_2 = p_1, \quad 2p_3 = p_0.$$

Substituting (3.17) into (3.16) we get

$$(3.18) \quad 3p_0 + 3p_1 = 1.$$

Substituting (3.17) into (3.13)

$$(3.19) \quad 3p_1 = p_0.$$

Substituting (3.19) into (3.18)

$$(3.20) \quad 4p_0 = 1,$$

and then

$$(3.21) \quad p_0 = p_7 = \frac{1}{4}, \quad p_1 = p_6 = \frac{1}{4 \cdot 3},$$

$$p_2 = p_5 = \frac{1}{4 \cdot 3 \cdot 2}, \quad p_3 = p_4 = \frac{1}{4 \cdot 2}.$$

Generalizing the previous computations we find a connection among the p 's.

Let A_j denote any sequence of j ($1 \leq j \leq r$) binary digits, that is let

$$(3.22) \quad A_j = i_1, i_2, \dots, i_j \quad (i_1, \dots, i_j \in \{0, 1\}).$$

We shall use also the notation

$$(3.23) \quad \bar{A}_j = \bar{i}_1, \bar{i}_2, \dots, \bar{i}_j.$$

Let $\alpha_r(A_r)$ denote the ergodic probability of the vector A_r in the case of r processable sequences.

At first we prove the following assertion.

Lemma 1. *If $r \geq 2$, $1 \leq k \leq r$, then*

$$(3.24) \quad k \alpha_r(00 \dots 001 A_{r-k}) = \alpha_r(00 \dots 000 A_{r-k}). \quad \square$$

According to this assertion the ergodic probability of a binary vector consisting of $(k-1)$ zeros, one 1 and any sequence A_{r-k} of length $(r-k)$ is k times smaller than the ergodic one of the binary vector in which k ones are continued by the sequence A_{r-k} .

Proof of Lemma 1.

For $k = 1$ we assert

$$(3.25) \quad \alpha_r(1 A_{r-1}) = \alpha_r(0 A_{r-1}),$$

which holds in consequence of the symmetry $\overline{1 A_{r-1}} = 0 \bar{A}_{r-1}$.

Let us suppose, (3.24) holds for $k = 1, \dots, s$, where $s < r$. Then we show that (3.25) holds for $k = s+1$, too.

Let us consider the binary vectors

$$(3.26) \quad 00 \dots 001 A_{r-s-1}.$$

We get this vector with a probability $\frac{2}{4}$ from itself, and with a probability $\frac{1}{4}$ from the vectors, whose first s elements contain punctually one 1, and the last $r-s$ elements are $1 A_{r-s-1}$.

Let B_j ($j = 1, \dots, s$) be the binary vector of length s , in which the j -th element is 1, the remaining elements are 0's.

Then

$$(3.27) \quad \begin{aligned} & \alpha_r(00 \dots 001 A_{r-s-1}) = \\ & = \frac{2}{4} \alpha_r(00 \dots 001 A_{r-s-1}) + \sum_{j=1}^s \frac{1}{4} \alpha_r(B_j 1 A_{r-s-1}), \end{aligned}$$

and from here

$$(3.28) \quad 2\alpha_r(00 \dots 001 A_{r-s-1}) = \sum_{j=1}^s \alpha_r(B_j 1 A_{r-s-1}).$$

Using twice the induction hypothesis we get for $j = 1, \dots, s-1$

$$(3.29) \quad \begin{aligned} & \alpha_r(B_j 1 A_{r-s-1}) = \alpha_r(00 \dots 010 \dots 01 A_{r-s-1}) = \\ & = \frac{1}{j} \alpha_r(00 \dots 001 \dots 10 A_{r-s-1}) = \\ & = \frac{1}{(j+1)j} \alpha_r(00 \dots 000 \dots 01 \overline{A_{r-s-1}}). \end{aligned}$$

For $j = s$ the induction hypothesis gives

$$(3.30) \quad \begin{aligned} & \alpha_r(00 \dots 000 \dots 11 A_{r-s-1}) = \\ & = \frac{1}{s} \alpha_r(00 \dots 000 \dots 00 \overline{A_{r-s-1}}). \end{aligned}$$

Let us substitute (3.29) and (3.30) into (3.28):

$$(3.31) \quad \begin{aligned} & \alpha_r(00 \dots 001 A_{r-s-1}) \left[2 - \sum_{j=1}^{s-1} \frac{1}{j(j+1)} \right] = \\ & = \frac{1}{s} \alpha_r(00 \dots 00 \overline{A_{r-s-1}}). \end{aligned}$$

Taking into account

$$(3.32) \quad \frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},$$

we get

$$(3.33) \quad \left(1 + \frac{1}{s}\right) \alpha_r(00 \dots 001 A_{r-s-1}) = \frac{1}{s} \alpha_r(00 \dots 00 \overline{A_{r-s-1}}),$$

that is

$$(3.34) \quad (s+1) \alpha_r(00 \dots 001 A_{r-s-1}) = \alpha_r(00 \dots 00 \overline{A_{r-s-1}}). \quad \square$$

Now we compute the ergodic probability of the vectors consisting of identical elements.

Lemma 2. *If $r \geq 2$ then*

$$(3.35) \quad \alpha_r(00 \dots 00) = \alpha_r(11 \dots 11) = \frac{1}{r+1}. \quad \square$$

Proof. We begin with the equality

$$(3.36) \quad \sum_{j=0}^{2^r-1} p_j = 1.$$

In consequence of the symmetry we have for $j = 2^{r-1}, \dots, 2^r - 1$

$$(3.37) \quad p_j = p_{2^r-1-j},$$

and therefore

$$(3.38) \quad \sum_{j=0}^{2^r-1} p_j = \sum_{j=0}^{2^{r-1}-1} p_j + \sum_{j=2^{r-1}}^{2^r-1} p_j = 2 \sum_{j=0}^{2^{r-1}-1} p_j.$$

We have got that for $k = 1$ holds

$$(3.39) \quad (k+1) \sum_{j=0}^{2^{r-k}-1} p_j = 1.$$

Let us suppose that (3.39) holds for $k = s$, where $s < r$. We show that than (3.39) holds for $k = s+1$ too.

By the induction hypothesis we have

$$(3.40) \quad (s+1) \sum_{j=0}^{2^{r-s}-1} p_j = 1.$$

We divide the numbers $0, 1, \dots, 2^{r-s}-1$ into two groups: $0, 1, \dots, \dots, 2^{r-s-1}-1$ and $2^{r-s-1}, \dots, 2^{r-s}-1$. The elements of the groups have the form $00 \dots 00 A_{r-s-1}$ and $00 \dots 01 \overline{A_{r-s-1}}$ resp. and we have a

one to one correspondence among them (the sum of the corresponding elements equal to $2^{r-s}-1$.) Therefore by Lemma 1 we get

$$(3.41) \quad (s+1) \sum_{j=0}^{2^{r-s}-1} p_j = (s+1) \sum_{j=0}^{2^{r-s-1}-1} p_j + (s+1) \sum_{j=2^{r-s-1}}^{2^{r-s}-1} p_j =$$

$$(3.42) \quad = (s+1) \sum_{j=0}^{2^{r-s-1}-1} p_j + (s+1) \sum_{j=0}^{2^{r-s-1}-1} \frac{p_j}{s+1} = (s+2) \sum_{j=0}^{2^{r-s-1}-1} p_j.$$

Hence (3.42) holds for every $k = 1, \dots, r$, among them for $k = r$, therefore

$$(3.43) \quad (r+1) p_0 = 1.$$

From (3.43) and the symmetry we get (3.35). \square

Now we can determine the ergodic probabilities.

Theorem 1. *Let us process the binary sequences $\xi_{i1}, \xi_{i2}, \dots, (i = 1, \dots, r)$ of independent random variables with common distribution*

$$(3.44) \quad P(\xi_{ij} = 0) = P(\xi_{ij} = 1) = \frac{1}{2} \quad (i = 1, \dots, r; j = 1, 2, \dots)$$

using the algorithm defined above. Then the ergodic probability of the vector (i_0, \dots, i_{r-1}) equals to

$$(3.45) \quad \alpha_r(i_0, \dots, i_{r-1}) = \frac{1}{(r+1) \prod_{\substack{2 \leq k \leq r \\ i_{k-1} \neq i_k}} k} \cdot \square$$

According to this theorem we get $\alpha_r(i_0, \dots, i_{r-1})$ dividing the basic value $\alpha_r(00 \dots 00) = \frac{1}{r+1}$ by the product of the indices of i 's differing from the previous one.

Proof of theorem 1. Let

$$(3.46) \quad j = \sum_{k=0}^{r-1} i_k \cdot 2^k$$

be the binary form of j for $j = 0, 1, \dots, 2^r - 1$. Due to the symmetry we have to deal only with the values $j = 0, 1, \dots, 2^{r-1} - 1$.

If $r = 1$, then a simple direct calculation gives $\alpha_1(0) = \frac{1}{2}$. Let now $r \geq 2$ be.

As for $j = 0$ the product in (3.45) is empty (i.e. there is no a change in the corresponding binary vector), Lemma 2. gives the same value, as (3.45).

Let now be s ($1 \leq s \leq r-1$) changes in j , that is let differ punctually the elements on the h_1 -th, \dots, h_s -th places from the elements preceding them.

Using Lemma 1 for $k = h_1, h_2, \dots, h_s$, we get

$$(3.47) \quad \alpha_r(j) \prod_{i=1}^s h_i = \alpha_r(0).$$

That was to be proved. \square

§. 4. On the processing speed

Processing binary sequences as before let the random variable η_{ij} denote the number of the processed elements of the i -th sequence in the j -th point of time.

According to the processing algorithm we have

$$(4.1) \quad 1 \leq \eta_{1j} \leq 2 \quad (j = 1, 2, \dots),$$

$$(4.2) \quad 0 \leq \eta_{ij} \leq 1 \quad (i = 2, \dots, r; j = 1, 2, \dots),$$

$$(4.3) \quad 1 \leq \sum_{i=1}^r \eta_{ij} \leq 2 \quad (j = 1, 2, \dots).$$

The processing speed $S^{(r)}$ is defined by

$$(4.4) \quad S^{(r)} = \lim_{t \rightarrow \infty} \frac{\sum_{j=1}^t M \left(\sum_{i=1}^r \eta_{ij} \right)}{t}.$$

There is a close connection between the distributions of σ_j and $\sum_{i=1}^r \eta_{ij}$:

$$(4.5) \quad P \left(\sum_{i=1}^r \eta_{ij} = 1 \right) = \frac{1}{2} P(\sigma_j = (0, 0, \dots, 0, 0)) + \\ + \frac{1}{2} P(\sigma_j = (1, 1, \dots, 1, 1)).$$

As $\sigma_1, \sigma_2, \dots$ is ergodic, the right side of (4.5) has a limit, whose value is $\frac{1}{r+1}$ according to Lemma 2. Therefore

$$(4.6) \quad \lim_{j \rightarrow \infty} M \left(\sum_{i=1}^r \eta_{ij} \right) = \frac{1}{r+1} \cdot 1 + \frac{r}{r+1} \cdot 2 = 2 - \frac{1}{r+1},$$

and from where

$$(4.7) \quad S^{(r)} = 2 - \frac{1}{r+1},$$

as the convergence of the sequence (4.6) implies the convergence (to the same limit) of the middle values of its elements in (4.4).

If we define the processing speed S_i for the i -th sequence by

$$(4.8) \quad S_1 = S^{(1)} \quad \text{and} \quad S_i = S^{(i)} - S^{(i-1)} \quad (i = 2, \dots, r),$$

then from (4.7) we get

$$(4.9) \quad S_1 = \frac{3}{2} \quad \text{and} \quad S_i = \frac{1}{i(i+1)} \quad (i = 2, \dots, r).$$

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