THE METHOD OF GIBBS—POOLE—STOCKMEYER IS NON-HEURISTIC

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Abstract

First we analyse the most popular pseudo-peripheral node finder method by N. Gibbs, W. Poole and P. Stockmeyer, which has been believed till now to be heuristic algorithm. Based on the properties of the nodes obtained by the Gibbs et als's method an exact definition is introduced for the pseudo-peripheral nodes, and some of their most important properties are also presented here. As an immediate consequence of the definition an important theorem is proved which serves as a theoretical foundation for both the Gibbs et als's method and for its different versions by A. George and J. Liu.

1. Introduction

In many application problems derived from an undirected, connected graph sometimes it is necessary to know the nodes at nearly largest distance from each other. The best known application area is the bandwidth/profile reduction problem [1], [5], [10], [11], [13]. However, during the last few years a number of important ordering algorithms [6], [7], [8], [10] has became known, published by Alan George and Joseph W—H Liu by which the overall efficiency can be achieved in treating sparse linear systems. For all these orderings such a kind of nodes are required at first.

There are a number of heuristic algorithms [9], [11], [12] for finding such nodes, from which the most popular one is by N. Gibbs, W. Poole and P. Stockmeyer [11]. For improving the efficiency of this so-called GPS method A. George and J. Liu described some modification strategies [9] from which the best one is used by the SPARSPAK [10].

In most of the tested cases both the GPS method and its modifications are working well, efficiently, but having no theoretical background there is no guarantee to find the required nodes, in fact.

In this paper we describe a theoretical foundation for the GPS method. First, a short discussion is presented for analysing the GPS method based on which an exact definition is introduced for the pseudo-peripheral nodes.

Summarizing their most important properties a theorem is proved which serves as a theoretical basis for both the GPS method and for its any modifications. This result apart from its own theoretical value has an importance in all the ordering algorithms mentioned before.

2. Some related definitions and notations

We use the notation $G=(X, E)$ for an undirected, connected graph; X is its node set, E is its edge set.

For describing the distance between two arbitrary nodes $x, y \in X$ we use the notation $d(x, y)$. Some of its most important properties are shown in [2].

Definition 2.1. [10] The eccentricity of an arbitrary node $x \in X$ is defined by

$$l(x)=\max_{y \in Y} d(x, y) .$$

So $l(x)$ denotes the maximal distance measured from x.

Definition 2.2. [10] The maximum value of eccentricities is called the diameter of the graph, so

$$\operatorname{diam}(G)=\max_{x \in X} l(x) .$$

The nodes of maximum eccentricity are called peripheral nodes.

Note, that if $x \in X$ is a peripheral node, then there exists a node $y \in X$ such that

$$d(x, y)=\operatorname{diam}(G)$$

is satisfied. Consequently, y is also a peripheral node and x and y are corresponding to each other.

A peripheral node x may have more than one peripheral node corresponding to it.

For level structures [1] rooted at $x \in X$ we use the notation introduced by A. George [10].

$$RLS(x)=\{L_0(x), L_1(x), \ldots, L_k(x)\},$$

where $k=l(x)$ and its last level is called its eccentricity level denoted by $L_{ec}(x)$. The nodes belonging to $L_{ec}(x)$ are the eccentricity nodes of x. The most important properties of $RLS(x)$ are shown in [2].

Remark 2.1.

For an arbitrary node $x \in X$

$$\left[\frac{\operatorname{diam}(G)}{2}\right] \leq l(x) \leq \operatorname{diam}(G)$$

relation is held.

For proof see for [2].

3. Short analysis of the GPS method [11]

As it is mentioned before, the GPS method is the first widely used algorithm for finding the so-called pseudo-peripheral nodes in the graph. It is a heuristic method so there is no guarantee to find exactly the nodes we wanted to get. Despite, as it is stated in [9], [11] in many tested cases it results a nearly optimal solution.

First of all, let us consider what is the pseudo-peripheral node, in fact. This term was introduced by A. George and J. Liu [9] where the authors followed Gibbs and the others [11] in saying "we use the pseudo-peripheral node to be one whose eccentricity is close" to the diameter. On the other hand, Gibbs et al. said in their work [11] that their algorithm is "to find the endpoints of a pseudo-diameter, that is, a pair of vertices that are at nearly maximal distance apart."

No doubt, the pseudo-peripheral nodes would be the same as the endpoints of the pseudo-diameter. However, no more exact description is given for any of these two important terms.

As it is known, the original GPS method [11] can be described as follows [9] (using George's notation).

Step 1. (Initialization)
Choose a starting node x which is a node of minimum degree.

Step 2. (Generation of level structure)
Construct a level structure rooted at x

$$RLS(x) = \{L_0(x), L_1(x), \ldots, L_{l(x)}(x)\}.$$

Step 3. (Sort the last level)
Sort the nodes in $L_{ec}(x)$ in the order of their increasing degree.

Step 4. (Test for termination)
For every $y \in L_{ec}(x)$, let us generate $RLS(y) = \{L_0(y), \ldots, L_{l(y)}(y)\}$. If $l(y) > l(x)$, then set $x \leftarrow y$ and go to step 3., otherwise start to treat the next node in $L_{ec}(x)$.

Step 5. (Exit)
The node x is the pseudo-peripheral node determined.

So using this algorithm the nodes x and $y \in L_{ec}(x)$ are the endpoints of the so-called pseudo-diameter denoted by pd; and $pd = d(x, y)$.

Let us see by an example in Figure 1 how the GPS method is working.

It can be seen very easily that nodes u and v in Figure 1 are peripheral nodes and

$$d(u, v) = \text{diam}(G) = 10.$$

Let us choose s_1 as a starting point for the GPS method. Then generating $RLS(s_1)$ its levels are shown in Figure 2.

So

$$L_{ec}(s_1) = \{t_1, t_2, t_3\}$$

is obtained, and its nodes are in correct order.

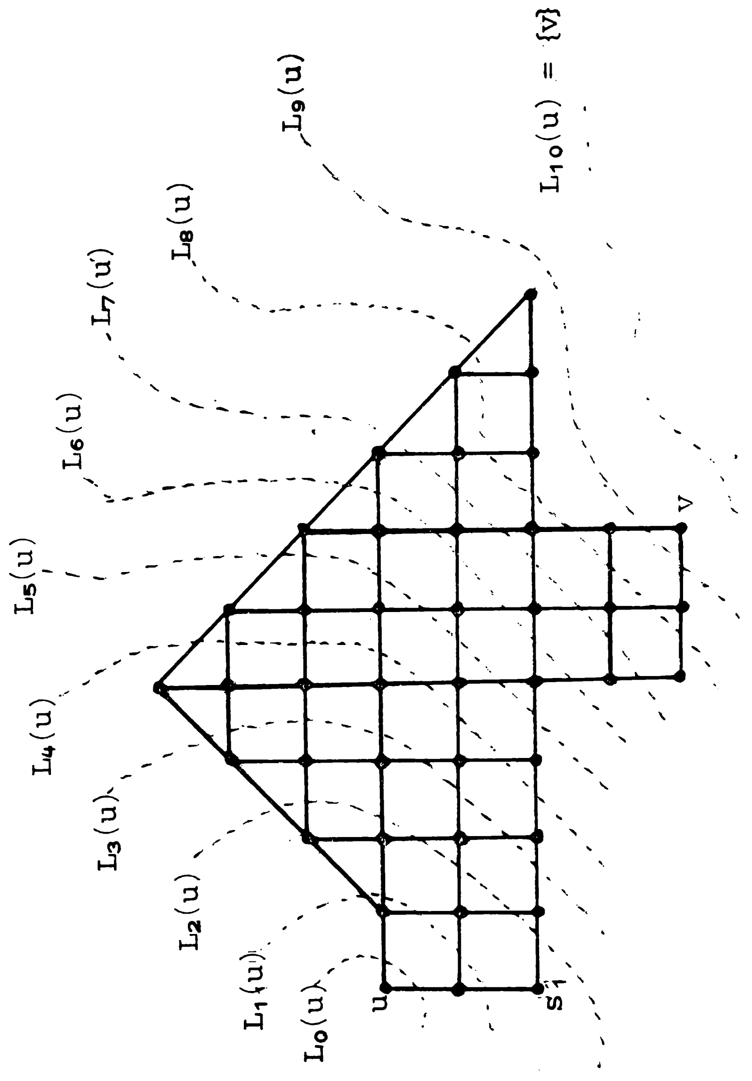

Figure 1.

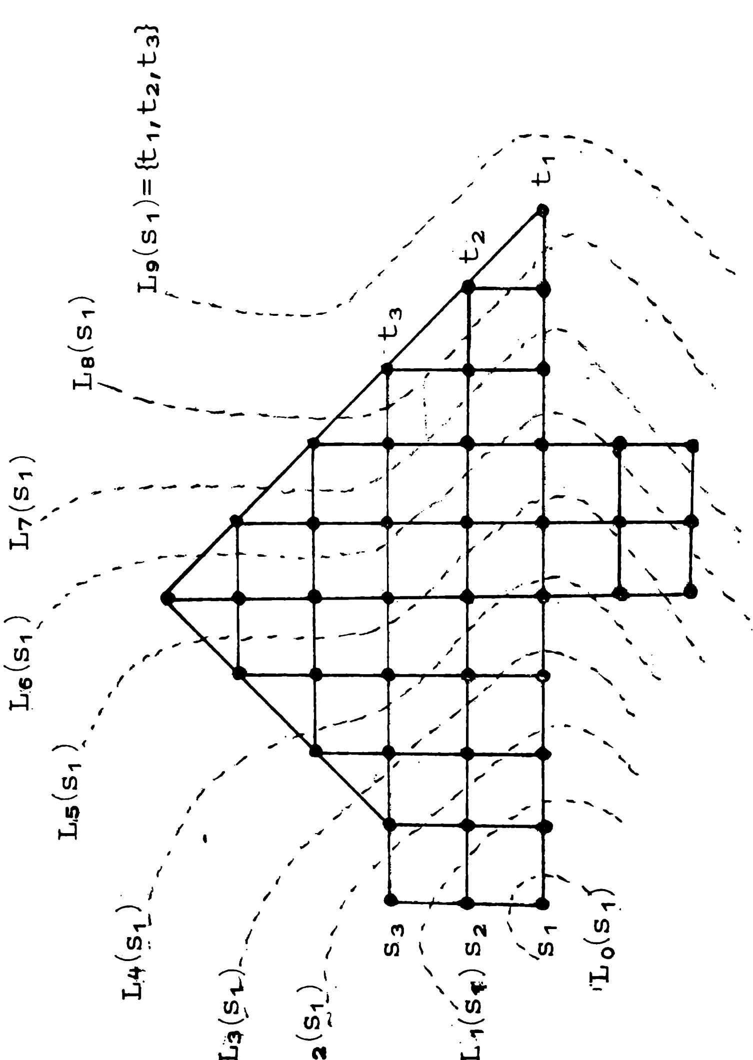
L9(s1)={t1,t2,t3}
L8(s1)
L7(s1)
L6(s1)
L5(s1)
L4(s1)
L3(s1)
L2(s1)
L1(s1)
L0(s1)
t1
t2
t3
s1
s2
s3

Figure 2.

Then generating $RLS(t_1)$ (by step 4)

$$L_{ec}(t_1) = \{s_1, s_2, s_3\} \quad \text{so} \quad l(t_1) = l(s_1)$$

follows.

It can be seen very easily that similar result is obtained when we generate either $RLS(t_2)$ or $RLS(t_3)$, namely, $L_{ec}(t_2) = \{s_1\}$; $l(t_2) = l(s_1)$ and $L_{ec}(t_3) = \{s_1\}$; $l(t_3) = l(s_1)$ are satisfied.

Consequently, s_1 and t_3 are the nodes obtained by the method ($d(s_1, t_3) = 9$ is the pseudo-diameter).

The above example is showing that the result node pair s_1, t_3 has quite similar properties as the peripheral nodes u and v. Here the only difference between them is in their eccetricities.

However, if t_1 is tested last, then s_1, t_1 are the result pair, and for $s_3 (= u) \in L_{ec}(t_1)$ $l(u) = 10$ is true. That is, for some $z \in L_{ec}(t_i)$ $(i = 1, 2, 3)$

$$l(z) > l(s_1)$$

may be satisfied.

4. Pseudo-peripheral nodes and its most important properties

As a result of the above observations the behavior of nodes obtained by the GPS method can remind us to the peripheral nodes. Let us define them as follows.

Definition 4.1 Let $x \in X$ be an arbitrary node. Let $y \in L_{ec}(x)$ be such an eccentricity point of x for which

$$l(y) = \max_{z \in L_{ec}(x)} l(z) \tag{4.1}$$

is satisfied.

The node x is a pseudo-peripheral node, if

$$l(x) = l(y) \tag{4.2}$$

is true.

The common value $l(x) = l(y)$ is the corresponding pseudo-diameter, and the nodes x and y are a pseudo-peripheral point pair corresponding to each other, x is starting and y is endpoint.

It is clear from the definition, that a peripheral node is such a kind of pseudo-peripheral node, whose eccentricity is the possible maximal one. So the peripheral nodes form a subset (but not necessarily real subset) in the set of pseudo-peripheral nodes. Figure 1 is showing an example when the peripheral node set is a real subset in the set of pseudo-peripheral nodes. On the other hand, for the $n \times m$ grid problem the two sets are equal so all its pseudo-peripheral nodes are also peripheral nodes. Consequently, the following trivial statement is true: in every graph there exist at least two pseudo-peripheral nodes in it.

The proof follows from the trivial statement by which for every graph (finite) there exist two nodes x and y for which

$$d(x, y) = \mathrm{diam}(G)$$

is satisfied.

Recall from [2] the following theorem.

Theorem 4.1

If node $x \in X$ is pseudo-peripheral node, then for every $z \in L_{ec}(x)$

$$l(z) = l(x)$$

is satisfied. □

(See Figure 1 for the illustration.)

As an immediate consequence of the definition 4.1 the following theorem is true.

Theorem 4.2

Let $x \in X$ be an arbitary node.

If x is not a pseudo-peripheral node, then there exists an eccentricity point of x, call it z, such that

$$l(z) > l(x)$$

relation is held. □

Proof. Let $x \in X$ be an arbitrary node. Then for its arbitrary eccentricity node $y \in L_{ec}(x)$

$$l(y) \geq l(x) \tag{4.4}$$

is a triviality.

By the assumption of the theorem, x is not a pseudo-peripheral node. Then by the definition 4.1

$$l(x) \neq \max_{y \in L_{ec}(x)} l(y) \tag{4.5}$$

follows. On the other hand, (4.4) is also true, consequently, there exists a node $z \in L_{ec}(x)$ such that

$$l(z) > l(x)$$

is satisfied. □

Conclusion. The GPS method, starting with an arbitrary node, converges to a pseudo-peripheral node in a finite number of steps.

Proof. Let $y_0 \in X$ be an arbitrary node which would be chosen as a starting node for the GPS method.

Then by applying the GPS method we are given a sequence of nodes in the following form:

$$y_i \in L_{ec}(y_{i-1}) \tag{4.6}$$

such that

$$l(y_i) > l(y_{i-1}) \tag{4.7}$$

relation is held for $i = 1, 2, \ldots, k$. (4.7) means that the series of eccentricities has to form a strongly increasing sequence. □

On the other hand, by the remark 2.1 for every $x \in X$

$$\left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor \le l(x) \le \operatorname{diam}(G) \tag{4.8}$$

relation is true, so in case of $x = y_0$ (4.8) has to be also satisfied. Consequently, starting from the very beginning, at most $\left\lceil \frac{\operatorname{diam}(G)}{2} \right\rceil$ steps are required for finding a node y_k for which

$$l(y_k) = pd \le \operatorname{diam}(G)$$

is satisfied for certain k.

So k is a finite number satisfying

$$1 \le k \le \left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor$$

relation.

We have to note, that in general case the relation

$$l(x) > \left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor$$

is held, so k has to be smaller than $\left\lfloor \frac{\operatorname{diam}(G)}{2} \right\rfloor$. In addition, as Gibbs et al said [11] in most of practical cases $k = 2$ was obtained.

By the theorem 4.2 and its conclusion a theoretical basis is assured that both the GPS method and its all versions have to result exactly a pair of pseudo-peripheral nodes. These methods are not already heuristic algorithms.

These results have a great importance in the ordering algorithms used when sparse linear systems are treated.

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