ON THE IMPROVING NEWTON'S METHOD FOR SOLVING NONLINEAR REAL EQUATIONS

by

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1. In [1] W. G. Hwang and J. Todd suggested some new points of view for the computation of the value $\sqrt{a}$ $(a>1)$. It is known that for the calculation of $\sqrt{a}$ we can use the classical Newton's method for the equation

$$f_1(x) := x^2 - a = 0.$$

In this case the iterative process

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

is given. In the above mentioned paper the authors recommend the

$$f_2(x) := 1 - \frac{a}{x^2} = 0$$

and the

$$f_3(x) := x^3 - ax = 0$$

equations, from which we get the following iterative processes:

$$x_{n+1} = \frac{x_n(3a - x_n^2)}{2a}$$

and

$$x_{n+1} = \frac{2x_n^3}{3x_n^2 - a}.$$

Considering we use the Newton's method, the order of convergence is quadratic.

It can be seen that the functions $f_2(x), f_3(x)$ mentioned above, can be written in the following forms:

$$f_2(x) := \frac{1}{x^2}(x^2 - a)$$

and

$$f_3(x) := x(x^2-a).$$

Now we try to substitute the functions $\frac{1}{x^2}$ and x by a suitably chosen auxiliary function $\varphi(x)>0$ so that

1. $\varphi(x) f(x) = 0$ should be equivalent to the equation $f(x) = 0$.

2. the error of one step should be as small as possible. Further on a class of functions, $\varphi(x, \lambda)$ will be taken. The Newton's method for the equation $\varphi(x, \lambda) f(x) = 0$ is

$$x_{n+1} = x_n - \frac{\varphi(x, \lambda) f(x)}{\varphi'_x(x, \lambda) f(x) - \varphi(x, \lambda) f(x)}. \tag{1.1}$$

Considering the requirements 1. and 2. and the (1.1) we suggest the class of functions

$$\varphi(x, \lambda) := e^{-\lambda x} \quad (\lambda>0).$$

We want to determine the parameter λ so that the difference $x_1 - \sqrt{a}$ should be as small as possible, where x_1 is the first value obtained by computation. Let's take the equation

$$e^{-\lambda x}(x^2-a) = 0 \quad (a>1) \tag{1.2}$$

and use the Newton's method, with any initial value $x_0>\sqrt{a}$ and approximate $\sqrt{a}$ by a monotone decreasing sequence. Suppose the relation

$$\lambda < \frac{2x}{x^2-a} \tag{1.3}$$

is valid. The error of the new approcimation x_1 will be minimal when the positive correction

$$\frac{x_0^2-a}{2x_0-\lambda(x_0^2-a)} \tag{1.4}$$

will be as large as possible.
Be

$$A(x, \lambda) := x - \frac{x^2-a}{2x-\lambda(x^2-a)} \tag{1.5}$$

an iterative function. From the Lagrange's formula we get

$$x_1-\sqrt{a} = A'_x(\xi, \lambda)\,(x_0-\sqrt{a}),$$

where $\sqrt{a}\leq\xi\leq x_0$. Supposing that

$$0<A'_x(x, \lambda)<1, \quad x\in[\sqrt{a}, x_0] \tag{1.6}$$

we get

$$\sqrt{a} < x_1 < x_0 ,$$

i.e. x_1 is a better approximation for the value $\sqrt{a}$ than x_0. From (1.6) and (1.3) we get

$$\lambda < \frac{2}{2x + \sqrt{2(x^2+a)}}, \quad x \in [\sqrt{a}, x_0] . \tag{1.7}$$

In order to make λ independent from x let us take its upper bound

$$\lambda_0 := \min_{x \in [\sqrt{a},\, x_0]} \frac{2}{2x + \sqrt{2(x^2+a)}} . \tag{1.8}$$

The expression defined by (1.8) is minimal in the interval $[\sqrt{a}, x_0]$ if $x = x_0$. Considering (1.4) we want to choose a "large" value for λ; if

$$0 < \lambda < \lambda_0$$

we propose

$$\lambda = \frac{1}{2x_0} < \lambda_0 \tag{1.9}$$

because it can be calculated very easily.
Returning to the solution of the equation (1.2) there is an obvious initial value x_0. The function being on the left side of (1.2) is convex in the interval $\left(-\infty, \frac{2-\sqrt{2+\lambda^2 a}}{\lambda}\right)$ if $\lambda < \frac{1}{\sqrt{a}}$. So the choice

$$x_0 := a \tag{1.10}$$

is a suitable initial value because

$$\bar{\lambda} = \frac{1}{2a} < \frac{1}{\sqrt{a}} . \tag{1.11}$$

2. In the first paragraph we dealt only with the numerical calculation of the root of the equation $f_1(x) := x^2 - a = 0$. Let's now consider the question more generally. Be x^* a simple root of a function $f(x)$ for which the relations

$$\begin{aligned} &f'(x) > 0 \\ &f''(x) > 0 \\ &(f')^2(x) - f''(x)\, f(x) > 0 \end{aligned} \tag{2.1}$$

are valid for $x \in [x^*, x_0]$. To solve the equation $f(x) = 0$ let's use the iterative method

$$x_{n+1} = A(x_n, \lambda) \tag{2.2}$$

generated by the function

(2.3) $$A(x,\lambda):=x-\frac{f(x)}{f(x)-\lambda f(x)},$$

where the relation

(2.4) $$0<\lambda<\frac{f'(x)}{f(x)},\quad x\in(x^*,x_0]$$

is valid for the parameter λ. We intend to construct a monoton decreasing sequence $\{x_n\}\to x^*$. A sufficient condition for the convergence of the method is given by the inequality

$$0<A'_x(x,\lambda)<1,$$

from which and from (2.4) we get

$$\lambda<\frac{f''(x)}{f'(x)+\sqrt{(f')^2(x)-f''(x)f(x)}},\quad x\in(x^*,x_0].$$

Let's denote

(2.5) $$\lambda_0=\inf_{x\in(x^*,\,x_0]}\frac{f''(x)}{f'(x)+\sqrt{(f')^2(x)-f''(x)f(x)}}$$

the upper bound for the values of the parameter λ. Considering that the error of the approximation $x_1^{(\lambda)}$ calculated by the method (2.2) will be small if the correction member

$$x_0-x_1^{(\lambda)}=\frac{f(x_0)}{f'(x_0)-\lambda f(x_0)}$$

is large, we have to choose the value λ near the bound λ_0. It can be seen that the method (2.2) is more efficient than the Newton's one where $\lambda=0$.
The order of the convergence of the method (2.2) is 2, because it is the Newton's method for the solution of the equation

$$e^{-\lambda x}f(x)=0.$$

1. Conclusion:

If the computation of the

(2.6) $$\bar{\lambda}:=\inf_{x\in(x^*,\,x_0]}\frac{f''(x)}{2f'(x)}$$

or

(2.7) $$\dot{\lambda}:=\frac{\delta}{2f'(x_0)},$$

when $f''(x) \geq \delta$, $x \in (x^* \; x_0]$, is not too complicated we suggest to use (2.2), method with the fix value $\bar{\lambda}_0(\bar{\lambda})$ for the solution of the equation $f(x) = 0$, wherever $f(x)$ satisfies the conditions (2.1). In this case the relation

$$x_i - x_{i+1}^{(\lambda)} \geq x_i - x_{i+1}$$

is valid step by step for the values $x_{i+1}^{(\lambda)}$, determined by the (2.2) and x_{i+1}, determined by the Newton's method. One multiplication and one subtraction at each step are the operational plus request of the suggested method.

2. Conclusion:

The choice of the parameter λ by (2.6) is interesting from theoretical point of view. We can observe, that using the function

(2.7) $$\lambda(x) := \frac{f''(x)}{2f'(x)}$$

instead of $\bar{\lambda}$ in the iterative method exactly the

(2.8) $$A(x) := x - \frac{f(x)}{f'(x) - \frac{1}{2}\frac{f''(x)f(x)}{f'(x)}}$$

iterative function is given. Varying the value of λ at each step according to (2.7) *we get, as a result of this optimizing efforts a third order convergent method,* the "osculatory" (in the literature usually "tangential") hiperbola method generated by (2.8) from the second order convergent Newton's method. It is remarkable *from the optimization point of view we can't find an optimal second order convergent method.*

3. In this paragraph we try to increase the convergence using a correction member in the numeral of the iterative function.
Be

(3.0) $$x_{n+1} = A(x_n, \mu)$$

generated by the function

(3.1) $$A(x, \mu) := x - \frac{f(x) + \mu f^2(x)}{f'(x)}$$

where the parameter μ is positive and the function $f(x)$ is characterized by the relations (2.1) in an environment of x^* a simple root of $f(x)$. We want to construct a monoton decreasing sequence $\{x_n\}$ converging to x^*. Using the inequalities

(3.2) $$0 < A'_x(x, \mu) < 1$$

to assure the convergence we get an upper bound for the parameter μ:

$$\mu < \frac{f''(x)}{2(f')^2(x) - f''(x) f(x)}, \quad x \in (x^*, x_0]. \tag{3.3}$$

In order to minimize the difference

$$x_1^{(\mu)} - x^* = \frac{f(x_0) + \mu f^2(x_0)}{f'(x_0)}$$

we have to choose a large value for the parameter μ.
Be

$$\mu_0 := \inf_{x \in (x^*, x_0]} \frac{f''(x)}{2(f')^2(x) - f''(x) f(x)} \tag{3.4}$$

and observing that the relation

$$\bar{\mu} < \mu_0$$

is true, where

$$\bar{\mu} := \inf_{x \in (x^*, x_0]} \frac{f''(x)}{2(f')^2(x)} \tag{3.5}$$

we suggest to choose the value $\bar{\mu}(\tilde{\mu})$ the iterative function (3.1), $\left(\text{where } \tilde{\mu} := \frac{\delta}{2(f')^2 x_0} \text{ with the same } \delta \text{ as in the 2. paragraph}\right)$.
Varying the value of parameter μ at each step by the function

$$\mu(x) := \frac{f''(x)}{2(f')^2(x)}$$

we get the iterative function

$$A(x) := x - \frac{f(x)}{f'(x)} - \frac{1}{2} \frac{f''(x) f(x)}{(f')^3(x)}. \tag{3.6}$$

It is known that (3.6) generates the method of "osculatory" parabolas. The iterative method generated by (3.1) can be regarded as the Newton's method for the equation of

$$\varphi(x, \mu) f(x) = 0$$

where

$$\varphi(x, \mu) := \frac{1}{1 + \mu f(x)}.$$

So we have the same conclusion from the point of view of the optimization as at the end of the paragraph 2.

Remarks:

1. The computer experiences show the efficiency of (2.2) and (3.0). The improving processes are advantageous especially in the starting period of the iteration process, where the values of $f(x)$ are large. If the initial value x_0 is too far from x^* then the improving process reduces the number of the steps of the iteration greatly.

2. Analysing the conditions (2.1) we get the relation

$$f(x) \leq \frac{1}{D} e^{\frac{x}{c}}$$

$(D > \emptyset;\ c \geq 1)$ for the function the root of which is iterated using the improving processes.

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