ON THE EMBEDDING OF A *pi*-AUTOMATON INTO AN *i*-AUTOMATON

by

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1. Preliminaries. In the papers [1, 2, 3] the general definition of an incompletely specified (or partial) finite automaton was proposed and some special classes of such automata were introduced. In this paper the following problems are solved. Let $\tilde{A}_{\text{gen}}$ be a partial finite *i*-automaton [1]. At first it is necessary to answer if there is any probabilitic automaton in $\tilde{A}_{\text{gen}}$. Secondly it is necessary to specify in correct form a partial finite *pi*-automaton which is contained in $\tilde{A}_{\text{gen}}$. A special case of this problems was investigated in [2, 3].

2. Definitions. First of all we recall some definitions of the paper [1]. Hereafter we use the term automaton to mean a finite automaton.

Let us use the following notations (where $\alpha, \beta \in \{0, 1\}$):

$$\Big|_{\alpha} = \begin{cases} (& \text{if } \alpha = 0 \\ [& \text{if } \alpha = 1 \end{cases}, \quad \Big|^{\beta} = \begin{cases}) & \text{if } \beta = 0 \\ ] & \text{if } \beta = 1 . \end{cases}$$

We also use the notations

$$\mathcal{R} = (-\infty, \infty),$$

$$\mathcal{R}^n = \{r \mid r = (r_1, r_2, \ldots, r_n),\ r_i \in \mathcal{R},\ i = 1, 2, \ldots, n\},$$

$$\mathcal{R}^{m,n} = \{R \mid R = (r_{ij})_{m,n},\ r_{ij} \in \mathcal{R},\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

for the sets of real numbers, vectors and matrices respectively, and the notations

$$\mathcal{P}^n = \Big\{r \mid r \in \mathcal{R}^n,\ r_i \in [0, 1],\ i = 1, 2, \ldots, n,\ \sum_i r_i = 1\Big\},$$

$$\mathcal{P}^{m,n} = \Big\{R \mid R \in \mathcal{R}^{m,n},\ r_{ij} \in [0, 1],\ \sum_j r_{ij} = 1,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\Big\}$$

for the sets of all probabilistic (or stochastic) n-dimensional vectors and mxn-matrices, respectively.

A partial vector with interval elements (an *i-vector)* is a subset of $\mathcal{R}^n$ defined as

$$\tilde{r} = \{|r \in \mathcal{R}^n,\ r_i \in \tilde{r}_i,\ i = 1, 2, \ldots, n\},$$

where

$$\tilde{r}_i = \underset{\sigma_i}{|} c_i, d_i \overset{\gamma_i}{|} \neq \emptyset,\ \sigma_i, \gamma_i \in \{0, 1\},\ c_i d_i \in \mathcal{R},$$

$$\sigma_i \gamma_i = 0 \Rightarrow c_i < d_i,\ \sigma_i \gamma_i = 1 \Rightarrow c_i \leq d_i.$$

An i-vector is specified in the form

$$\tilde{r} = \left(\underset{\sigma_1}{|} c_1, d_1 \overset{\gamma_1}{|},\ \underset{\sigma_2}{|} c_2, d_2 \overset{\gamma_2}{|}, \ldots, \underset{\sigma_n}{|} c_n, d_n \overset{\gamma_n}{|} \right). \tag{1}$$

Accordingly, an *i-matrix* is a subset of $R^{m,n}$ defined as

$$\tilde{R} = \{R | R \in \mathcal{R}^{m,n},\ r_{ij} \in \tilde{r}_{ij},\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\},$$

where

$$\tilde{r}_{ij} = \underset{\sigma_{ij}}{|} c_{ij}, d_{ij} \overset{\gamma_{ij}}{|} \neq \emptyset,\ \sigma_{ij}, \gamma_{ij} \in \{0, 1\},\ c_{ij}, d_{ij} \in \mathcal{R},$$

$$\sigma_{ij} \gamma_{ij} = 0 \Rightarrow c_{ij} < d_{ij},\ \sigma_{ij} \gamma_{ij} = 1 \Rightarrow c_{ij} \leq d_{ij}.$$

An i-matrix is specified in the form

$$\tilde{R} = \left(\underset{\sigma_{ij}}{|} c_{ij}, d_{ij} \overset{\gamma_{ij}}{|} \right)_{m,n}.$$

For example,

$$\tilde{R} = \begin{pmatrix} \left[\frac{1}{3}, 2\frac{1}{4}\right) & [-3, 1] \\ \left(-\frac{1}{6}, \frac{1}{2}\right) & \left(\frac{2}{3}, 5\right] \end{pmatrix}.$$

Let $X = \{X_1, X_2, \ldots, X_n\}$, $Y = \{Y_1, Y_2, \ldots, Y_k\}$, $A = \{A_1, A_2, \ldots, A_m\}$ be the alphabets of inputs, outputs and states, respectively. Then a *partial generalized i-automaton* (briefly, an *i-automaton)* is a system

$$\tilde{A}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle, \tag{2}$$

where

$$\tilde{r}^{(0)} = \left(\underset{\sigma_1}{|} c_1 d_1 \overset{\gamma_1}{|},\ \underset{\sigma_2}{|} c_2, d_2 \overset{\gamma_2}{|}, \ldots, \underset{\sigma_m}{|} c_m, d_m \overset{\gamma_m}{|} \right)$$

is an initial i-vector and

$$\tilde{R} = \left({}_{\sigma_{si,lj}}| \; c_{si,lj}, d_{si,lj} \; |^{\gamma_{si,lj}} \right)_{nm,\,km}$$

is a transition – output i-matrix.

An i-automaton (2) defines a set of completely specified generalized automata such that

$$A_{\text{gen}} \in \tilde{A}_{\text{gen}} \Leftrightarrow r^{(0)} \in \tilde{r}^{(0)} \,\&\, R \in \tilde{R}$$

where $A_{\text{gen}} = \langle X, A, Y, r^{(0)}, R \rangle$.

A partial probabilistic vector with interval elements (a pi-vector) is a subset of $\mathcal{D}^n$ defined as

$$\tilde{p} = \left\{ r \mid r \in \mathcal{D}^n, \quad r_i \in \tilde{r}_i, \quad i = 1, 2, \ldots, n, \quad \sum_i r_i = 1 \right\}$$

where

$$\tilde{r}_i = {}_{\alpha_i}| \; a_i, b_i \; |^{\beta_i} \subseteq [0, 1], \quad \tilde{r}_i \neq \emptyset \quad \text{for} \quad i = 1, 2, \ldots, n.$$

A pi-vector is specified in the form

$$\tilde{p} = \left({}_{\alpha_1}| \, a_1, b_1 \, |^{\beta_1}, \; {}_{\alpha_1}| \, a_2, b_2 \, |^{\beta_2}, \ldots, {}_{\alpha_n}| \, a_n, b_n \, |^{\beta_n} \right) \tag{3}$$

where the obvious condition $\sum r_i = 1$ is omitted.

We say that a pi-vector is *correctly specified* if for each $r_j \in {}_{\alpha_j}| \, a_j, b_j \, |^{\beta_j}$ there are $r_i \in {}_{\alpha_i}| \, a_i, b_i \, |^{\beta_i}$ $i \neq j$ such that $\sum_{s=1}^{n} r_s = 1$.

A subset of $\mathcal{D}^{m,n}$ defined as

$$\tilde{P} = \left\{ P \mid P \in \mathcal{D}^{m,n}, \quad r_{ij} \in \tilde{r}_{ij}, \quad \sum_j r_{ij} = 1, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n \right\}$$

where

$$\tilde{r}_{ij} = {}_{\alpha_{ij}}| \; a_{ij}, b_{ij} \; |^{\beta_{ij}} \subseteq [0, 1], \quad \tilde{r}_{ij} \neq \emptyset, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

is called a *pi-matrix* and is specified in the form where the obvious conditions $\sum_j r_{ij} = 1$ are omitted. A pi-matrix is *correctly specified* if each of its rows is a correctly specified pi-vector.

For example,

$$\tilde{P} = \begin{pmatrix} [0{,}3;\ 0{,}6) & (0{,}1;\ 0{,}2] & (0{,}3;\ 0{,}5] \\ [0{,}5;\ 0{,}6] & 0{,}2 & [0{,}2;\ 0{,}3] \\ [0{,}2;\ 0{,}4] & [0;\ 0{,}3) & (0{,}3;\ 0{,}8] \end{pmatrix}$$

is a correctly specified square pi-matrix of order *3*.

A *partial pi-automaton* (briefly, *pi-automaton)* is a system

(4) $$\tilde{A}_{pr} = \langle X, A, Y, \tilde{p}^{(0)}, \tilde{P} \rangle$$

where $\tilde{r}^{(0)}$ is a correctly specified m-dimensional pi-vector (a partial initial probabilistic distribution on the state set A) and $\tilde{p}$ is a correctly specified pi-matrix of size $nm \times km$ (a partial transition-output probability matrix). A partial pi-automaton (4) defines a set of completely specified probabilistic automata such that

$$A_{pr} = \langle X, A, Y, \tilde{p}^{(0)}\tilde{P}, \rangle \in A_{pr} \Leftrightarrow p^{(0)} \in \tilde{p}^{(0)} \,\&\, P \in \tilde{P}.$$

3. The problem. Now let us formulate the main problem of this paper. Let $\tilde{A}_{\text{gen}}$ be an i-automaton (2). At first it is necessary to answer the question if there is any probabilistic automaton in the set $\tilde{A}_{\text{gen}}$ or not. And then it is necessary to find a correctly specified pi-automaton $\tilde{A}_{pr}$ such that $\tilde{A}_{pr} \subseteq \tilde{A}_{\text{gen}}$ and no probabilistic automaton belongs to the set $\tilde{A}_{\text{gen}} \setminus \tilde{A}_{pr}$. It is clear that for the solution of this problem it is sufficient to solve an analogous problem for an i-vector and a pi-vector.

4. The conditions of correct specification. In the paper [1] the following theorem was proved.

Theorem 1. *Let $\tilde{p}$ be a pi-vector* (3). *Then $\tilde{p}$ is correctly specified if and only if the following conditions hold for $j = 1, \ldots, n$:*

(5) $$a_j \geq 1 - \sum_{i \neq j} b_i$$

and

(6) $$a_j = 1 - \sum_{i \neq j} b_i \,\&\, \exists_i : i \neq j, \;\; \beta_i = 0 \Rightarrow \alpha_j = 0;$$

(a)

(7) $$b_j \leq 1 - \sum_{i \neq j} a_i$$

and

(8) $$b_j = 1 - \sum_{i \neq j} a_i \,\&\, \exists_i : i \neq j, \;\; \alpha_i = 0 \Rightarrow \beta_j = 0.$$

(b) ☒

This theorem makes it possible to answer the question whether a pi-vector $\tilde{p}$ is correctly specified or not.

5. The solution. The following two theorems give the solution ot the problems formulated above.

Theorem 2. *Let $\tilde{r}$ be an i-vector* (1).
Let c_i', σ_i' be defined as

$$c_i' = \begin{cases} c_i & \text{if } c_i \geq 0 \\ 0 & \text{if } c_i < 0 \end{cases} \qquad \sigma_i' = \begin{cases} \sigma_i & \text{if } c_i \geq 0 \\ 1 & \text{if } c_i < 0. \end{cases}$$

Then $\tilde{r}\cap\mathcal{P}^n\neq\emptyset$ *if and only if the following conditions hold:*

(a′) $$\sum_i c_i'\leq 1 \quad \text{and} \quad \sum_i c_i' = 1\Rightarrow \underset{i}{\&}\,\sigma_i' = 1$$

(b′) $$d_i\geq 0;\quad i = 1,\dots,n;\quad \sum_i d_i\geq 1 \quad \text{and} \quad \sum_i d_i = 1\Rightarrow \underset{i}{\&}\,\gamma_i = 1\,. \boxtimes$$

Proof. For the proof of necessity let $p = (p_1, p_2, \dots, p_n)$ be a probabilistic vector ($p\in\mathcal{P}^n$) such that $p\in\tilde{r}$. Then $p_i\in[0, 1]$, $\sum_i p_i = 1$ and for every i

$$p_i\in {}_{\sigma_i}|\, c_i', d_i\,|^{\gamma_i}\subseteq {}_{\sigma_i}|\, c_i, d_i\,|^{\gamma_i}\,.$$

This implies that

$$[0, 1]\cap {}_{\sigma_i}|\, c_i', d_i\,|^{\gamma_i}\neq\emptyset\,,\quad i = 1,\dots,n\,,$$

(9)

$$\sum p_i = 1\in {}_{\sigma}|c_i', \sum d_i|^{\gamma}\,,$$

where $\sigma = \&\sigma_i'$, $\gamma = \&\,\gamma_i$. The necessity of conditions (a′) and (b′) obviously follows from (9).

Conversely, assume that conditions (a′) and (b′) hold for $\tilde{r}$. We prove the existence of a probabilistic vector p such that $p\in\tilde{r}\cap\mathcal{P}^n$. From (a′) and (b′) we have that if $d_i = c_i'$ for $i = 1, \dots, n$ then $\sum_i d_i = \sum_i c_i' = 1$, $\underset{i}{\&}\,\sigma_i' = \underset{i}{\&}\,\gamma_i = 1$. Thus if $p_i = d_i$ for $i = 1, \dots, n$ then $p\in\tilde{r}\cap\mathcal{P}^n$. If there is an h such that $d_h > c_h'$ then $\sum_i(d_i-c_i')>0$ and we take the following elements of the vector p

(10) $$p_j = c_j'+\frac{1-\sum_i c_i'}{\sum_i(d_i-c_i')}(d_j-c_j')\,,\quad j = 1,\dots,n\,.$$

From (a′), (b′) and (10) we have that

$$p_j\in {}_{\sigma_j'}|\, c_j', d_j\,|^{\gamma_j}\subseteq {}_{\sigma_j}|\, c_j, d_j\,|^{\gamma_j};\quad j = 1,\dots,n;\quad \sum p_j = 1\,,$$

i.e. $p\in\tilde{r}$ and $p\in\mathcal{P}^n$. This completes the proof of Theorem 2. $\boxtimes$

Theorem 3. *Let* $\tilde{r}$ *be an i-vector* (1) *such that conditions* (a') *and* (b') *hold. Let* $\tilde{r}'$ *be an i-vector defined as*

(11) $$\tilde{r}' = \left({}_{\delta_1}|\, u_1, v_1\,|^{\varepsilon_1}, {}_{\delta_2}|\, u_2, v_2\,|^{\varepsilon_2}, \dots, {}_{\delta_n}|\, u_n, v_n\,|^{\varepsilon_n}\right),$$

where

(12) $${}_{\delta_i}|\, u_i, v_i\,|^{\varepsilon_i} = {}_{\sigma_i}|\, c_i, d_i\,|^{\gamma_i}\cap[0, 1]\quad i = 1,\dots,n\,.$$

Then a pi-vector (3) *is correctly specified and* $\tilde{p} = \tilde{r} \cap \mathcal{P}^n$ *if and only if the following conditions hold for* $j = 1, \ldots, n$:

$$b_j = \min\Big(v_j, 1 - \sum_{i \neq j} u_i\Big), \tag{13}$$

$$a_j = \max\Big(u_j, 1 - \sum_{i \neq j} b_i\Big), \tag{14}$$

$$\beta_j = 0 \Leftrightarrow (b_j = v_j) \& (\varepsilon_j = 0) \vee \Big(b_j = 1 - \sum_{i \neq j} u_i\Big) \& \exists\, i \neq j : \delta_i = 0\,, \tag{15}$$

$$\alpha_j = 0 \Leftrightarrow (a_j = u_j) \& (\delta_j = 0) \vee \Big(a_j = 1 - \sum_{i \neq j} b_i\Big) \& \exists\, i \neq j : \beta_i = 0\,. \tag{16}$$ ☒

Proof. Since conditions (a′) and (b′) hold for $\tilde{r}$, thus $\tilde{r} \cap \mathcal{P}^n \neq \emptyset$. In accordance with (11) and (12), $\tilde{r}' \subseteq \tilde{r}$ and $\tilde{r}' \cap \mathcal{P}^n = \tilde{r} \cap \mathcal{P}^n$. Then conditions (a′) and (b′) hold for $\tilde{r}'$ too, i.e.

$$\sum_i u_i \leq 1 \quad \text{and} \quad \sum_i u_i = 1 \Rightarrow \underset{i}{\&}\, \delta_i = 1\,,$$

$$\sum_i v_i \geq 1 \quad \text{and} \quad \sum_i v_i = 1 \Rightarrow \underset{i}{\&}\, \varepsilon_i - 1\,.$$

Let $\tilde{p}$ be defined as in (3) and (13) – (16) and $p = (p_1, p_2, \ldots, p_n)$ be a probabilistic vector ($p \in \mathcal{P}^n$) such that $p \in \tilde{r}'$. Then for the vector p $\sum_i p_i = 1$ and

$$p_i \in {}_{\delta_i}|\, u_i, v_i\,|^{\varepsilon_i} \subseteq [0, 1]\,, \quad i = 1, \ldots, n \tag{17}$$

holds. This implies that

$$p_j = \begin{cases} \leq v_j & \text{if} \quad \varepsilon_j = 1\,, \\ < v_j & \text{if} \quad \varepsilon_j = 0 \end{cases} \tag{18}$$

and

$$p_j = 1 - \sum_{i \neq j} p_i = \begin{cases} \leq 1 - \sum_{i \neq j} u_i & \text{if} \quad \underset{i \neq j}{\&}\, \delta_i = 1\,, \\ < 1 - \sum_{i \neq j} u_i & \text{if} \quad \exists\, i : i \neq j, \quad \delta_i = 0\,. \end{cases} \tag{19}$$

Then in accordance with (18), (19), (13) and (15)

$$\begin{aligned} \beta_j = 0 \Rightarrow p_j < b_j = \min\Big(v_j, 1 - \sum_{i \neq j} u_i\Big), \\ \beta_j = 1 \Rightarrow p_j \leq b_j = \min\Big(v_j, 1 - \sum_{i \neq j} u_i\Big). \end{aligned} \tag{20}$$

From (17) we have also that

$$p_j = \begin{cases} \geq u_j & \text{if} \quad \delta_j = 1\,, \\ > u_j & \text{if} \quad \delta_j = 0 \end{cases} \tag{21}$$

and in accordance with (20)

$$p_j = 1 - \sum_{i \neq j} p_i = \begin{cases} > 1 - \sum\limits_{i \neq j} b_j & \text{if } \exists i : i \neq j, \beta_i = 0\,, \\ \geq 1 - \sum\limits_{i \neq j} b_i & \text{if } \underset{i \neq j}{\&} \beta_i = 1\,. \end{cases} \tag{22}$$

It follows from (21), (22) and (14), (16) that

$$\begin{aligned} \alpha_j = 0 &\Rightarrow p_j > a_j = \max\Big(u_j, 1 - \sum_{i \neq j} b_i\Big)\,, \\ \alpha_j = 1 &\Rightarrow p_j \geq a_j = \max\Big(u_j, 1 - \sum_{i \neq j} b_i\Big)\,. \end{aligned} \tag{23}$$

Then in accordance with (20), (23) $p_j \in {}_{\alpha_j}|a_j, b_j|^{\beta_j}$ $j = 1, \ldots, n$, i.e. $p \in \tilde{r}' \Rightarrow p \in \tilde{p}$ and therefore for $\tilde{p}$ the conditions (a′) and (b′) hold.

Conversely, assume that $p \in \tilde{p}$, where $\tilde{p}$ is defined as in (3) and (13)–(16). It follows from (13)–(16) that ${}_{\alpha_i}|a_i, b_i|^{\beta_i} \subseteq {}_{\delta_i}|u_i, v_i|^{\varepsilon_i}$ for $i = 1, \ldots, n$ and $\tilde{p} \subseteq \tilde{r}'$.

Therefore for every $p \in \mathcal{D}^n$

$$p \in \tilde{r}' \Leftrightarrow p \in \dot{p}$$

and

$$\tilde{p} = \tilde{r}' \cap \mathcal{D}^n = \tilde{r} \cap \mathcal{D}^n\,.$$

Now it is necessary to prove that if a *pi*-vector $\tilde{p}$ is defined as in (3) and (13)–(16) then it is correctly specified, i.e. that for $\tilde{p}$ conditions (a) and (b) hold. From (14) and (16) we have that for $\tilde{p}$ (5) and (6) hold. Therefore it is necessary to prove only that for $\tilde{p}$ (7) and (8) hold.

Assume that

$$a_j = \begin{cases} u_j, & j \neq j_1, j_2, \ldots, j_k, \\ 1 - \sum\limits_{i \neq j} b_i, & j = j_1, j_2, \ldots, j_k\,, \end{cases} \tag{24}$$

and consider the case $j \neq j_\nu$; $\nu = 1, \ldots, k$.

In accordance with (24)

$$\sum_{i \neq j} a_i = \sum_{\substack{i \neq j, j_\nu \\ \nu = 1, \ldots, k}} u_i + k - \sum_{\nu = 1, \ldots, k} \sum_{j \neq j_\nu} b_i$$

and since

$$\sum_{i \neq j_k} b_i = \sum_{\nu = 1, \ldots, k-1} b_{j_\nu} + \sum_{\substack{i \neq j, j_\nu \\ \nu = 1, \ldots, k}} b_i + b_j\,,$$

thus

$$1 - \sum_{i \neq j} a_i = \sum_{\substack{i \neq j, j_\nu \\ \nu = 1, \ldots, k}} (b_i - u_i) + (k-1)\Big(\sum_i b_i - 1\Big) + b_j\,. \tag{25}$$

If $k=0$ then from (25), (13) we have $1-\sum_{i\neq j} a_j = 1-\sum_{i\neq j} u_i \geq b_j$. Since $b_i \geq a_i \geq u_i$ for $i=1,\dots,n$ and $\sum_i b_i \geq 1$, thus for $k\geq 1$ we have $1-\sum_{i\neq j} a_i \geq b_i$. Therefore condition (7) holds for $j\neq j_\nu$; $\nu=1,\dots,k$.

Let j be now such that $j=j_\xi$, $\xi\in\{1,2,\dots,k\}$ then

$$\sum_{i\neq j_\xi} a_i = \sum_{\substack{i\neq j_\nu,\\ \nu=1,\dots,k}} u_i + k - 1 - \sum_{\nu=1,\dots,k-1}\ \sum_{i\neq j_\nu} b_i - \sum_{\nu=\xi+1,\dots,k}\ \sum_{i\neq j_\nu} b_i$$

and since

$$\sum_{i\neq j_{\xi-1}} b_i = \sum_{\nu=1,\dots,\xi-2} b_{j_\nu} + \sum_{\nu=\xi+1,\dots,k} b_{j_\nu} + \sum_{\substack{i\neq j_\nu\\ \nu=1,\dots,k}} b_i + b_{j_\xi}$$

thus

$$1-\sum_{i\neq j_\xi} a_i = \sum_{\substack{i\neq j_\nu\\ \nu=1,\dots,k}} (b_i-u_i) + (k-2)\Big(\sum_i b_i - 1\Big) + b_{j_\xi}. \tag{26}$$

If $k=1$ then in accordance with (26) and (13) we have

$$1-\sum_{i\neq j_1} a_i = \sum_{i\neq j_1}(b_i-u_i) + \Big(\sum_i b_i - 1\Big) + b_{j_1} = 1-\sum_{i\neq j_1} u_i \geq b_{j_1}.$$

Since $b_i\geq a_i\geq u_i$ for $i=1,\dots,n$ and $\sum_i b_i\geq 1$ thus for $k\geq 2$ we have $1-\sum_{i\neq j_\xi} a_i \geq$ $\geq b_{j_\xi}$. Therefore condition (7) holds for $j=j_\xi$; $\xi=1,\dots,k$ also.

Now we prove that for $\tilde{p}$, condition (8) holds. Assume that

$$b_j = 1-\sum_{i\neq j} a_i\,,\quad \alpha_{i_1}=0\,,\quad i_1\neq j \tag{27}$$

and prove that in this case $\beta_j=0$.

Firstly consider the case $j\neq j_\nu$; $\nu=1,\dots,k$. If $k=0$ then from (25) and (27) we have $b_j = 1-\sum_{i\neq j} a_i = 1-\sum_{i\neq j} u_i$. If $\delta_{i_1}=0$ then it is follows from (15) that $\beta_j=0$. Assume now that $\delta_{i_1}=1$. Since $\alpha_{i_1}=0$ thus it follows from (16) and (24) that $a_{i_1}=u_{i_1}=1-\sum_{i\neq i_1} b_i$ and there must be an i such that $i\neq i_1$ and $\beta_i=0$. But in this case $b_j = 1-\sum_{i\neq j} u_i = \sum_{i\neq i_1} b_i - \sum_{i\neq j, i_1} u_i$. This implies that $b_i=a_i=u_i$ for $i\neq i_1$ and since $\tilde{p}\neq\emptyset$ thus there is no i such that $i\neq i_1$, $\beta_i=0$ and $\alpha_{i_1}=1$. But this contradicts our assumption. Therefore $\delta_{i_1}=0$ and $\beta_j=0$.

If $k\geq 1$, $b_j\geq 1-\sum_{i\neq j} a_i$ then from (25) we have

$$\sum_{\substack{i\neq j, j_\nu\\ \nu=1,\dots,k}} (b_i-u_i) + (k-1)\Big(\sum_i b_i - 1\Big) = 0\,. \tag{28}$$

If $k = 1$ then $b_i = u_i = a_i$ for $i \neq j, j_1$. Therefore $\alpha_i = \beta_i = 1$, $i \neq j, j_1$, and then $i_1 = j_1$, $\alpha_{j_1} = 0$. In this case from (27) we have $a_{j_1} = 1 - \sum_{i \neq j_1} b_i$. Since $a_{j_1} = 1 - \sum_{i \neq j_1} b_i > u_j$ thus in accordance with (16)

$$\alpha_{j_1} = 0 \Leftrightarrow \exists i : i \neq j_1 \quad \text{and} \quad \beta_i = 0 ,$$

and if $\alpha_{j_1} = 0$, $\beta_i = 1$ and $i \neq j, j_1$ then $\beta_j = 0$. For $k > 1$ equation (28) is true only if $\sum_i b_i = 1$. Then $a_i = b_i$ and $\alpha_i = \beta_i = 1$ for all i, i.e. there is no i_1 such that $\alpha_{i_1} = 0$. Thus we have proved that condition (8) holds for $j \neq j_\nu$; $\nu = 1, \ldots, k$.

Let now $j = j_\xi$, $\xi \in \{1, 2, \ldots, k\}$, then for $k = 1$ it follows from (26) and (27) that $b_{j_1} = 1 - \sum_{i \neq j_1} u_i$, and this case is analogous to the case $j \neq j_\nu$; $\nu = 1, \ldots, k$; $k = 0$ which we have already investigated above.

If $k \geq 2$, $b_{j_\xi} = 1 - \sum_{i \neq j_\xi} a_i$ then from (26) we have

$$\sum_{\substack{i \neq j_\nu, \\ \nu = 1, \ldots, k}} (b_i - u_i) + (k-2)\left(\sum_i b_i - 1\right) = 0 . \tag{29}$$

If $k = 2$ then $b_i = u_i = a_i$ for $i \neq j_1, j_2$. This case is analogous with the case $j \neq j_\nu$; $\nu = 1, \ldots, k$; $k = 1$ which have also been investigated above. And at last for $k > 2$, equation (29) is true only if $\sum_i b_i = 1$ and this case is analogous with the case $j \neq j_\nu$; $\nu = 1, \ldots, k$; $k > 1$. Thus we have proved that for $\tilde{p}$, condition (8) also holds and therefore a *pi*-vector defined as in (3) and (13)–(16) is correctly specified.

After all we notice that for any correctly specified *pi*-vector $\tilde{p}'$, such that $\tilde{p}' \neq \tilde{p}$ the conditions of Theorem 3 do not hold as there is a vector $p (\in \mathcal{P}^n)$ such that either $p \in \tilde{p}'$ and $p \bar{\in} \tilde{p}$ (and therefore $p \bar{\in} \tilde{r}$) or $p \in \tilde{p} \subseteq \tilde{r}$ and $p \bar{\in} \tilde{p}'$. This completes the proof of Theorem 3. ⊠

6. Correction operation. Any *i*-vector (1) such that $\tilde{r} \cap \mathcal{P}^n \neq \emptyset$ may be treated as an incorrectly specified *pi*-vector. So the procedure for constructing the *pi*-vector $\tilde{p} = \tilde{r} \cap \mathcal{P}^n$ may be called a *correction operation* (in notation $\tilde{p} = Cor\, \tilde{r}$). In accordance with Theorems 2,3 this procedure consists of the following steps:

1. Examine if for $\tilde{r}$, conditions (a′) and (b′) hold (i.e. if $\tilde{r} \cap \mathcal{P}^n \neq \emptyset$).
2. If $\tilde{r} \cap \mathcal{P}^n \neq \emptyset$ then construct the *i*-vector $\tilde{r}'$, in accordance with (11) and (12).
3. Find b_j and β_j for all $j = 1, \ldots, n$ in accordance with (13) and (15).
4. Find a_j and α_j for all $j = 1, \ldots, n$ in accordance with (14) and (16).

This completes the construction of $\tilde{p} = Cor\, \tilde{r}$.

Example. Let $\tilde{r}$ be an *i*-vector defined as

$$\tilde{r} = ([-1{,}2;\ 5{,}1],\ (0{,}2;\ 1{,}2),\ (-0{,}3;\ 0{,}2],\ [0{,}1;\ 1),\ (0{,}4;\ 0{,}5])$$

and it is necessary to find $\tilde{p} = Cor\ \tilde{r}$. For $\tilde{r}$, conditions (a') and (b') hold as $\sum_i c'_i = 0{,}7$ and $\sum_i d. = 8$. In accordance with (11) and (12) we find

$$\tilde{r}' = ([0;\ 1], (0{,}2;\ 1], [0;\ 0{,}2], [0{,}1;\ 1), (0{,}4;\ 0{,}5]).$$

Now from (13)–(16) we have

$$\tilde{p} = ([0; 0{,}3), (0{,}2; 0{,}5), [0; 0{,}2], [0{,}1; 0{,}4), (0{,}4; 0{,}5]).$$

Let $\tilde{R}$ be an i-matrix defined as

$$\tilde{R} = \begin{pmatrix} \tilde{r}^{(1)} \\ \tilde{r}^{(2)} \\ \vdots \\ \tilde{r}^{(m)} \end{pmatrix},$$

where $\tilde{r}^{(i)} \in \mathcal{R}^n$, $i = 1, \ldots, m$. Then for $\tilde{R}$ a correction operation is defined as

$$\tilde{P} = Cor\,\tilde{R} = \begin{pmatrix} Cor\,\tilde{r}^{(1)} \\ Cor\,\tilde{r}^{(2)} \\ \vdots \\ Cor\,\tilde{r}^{(m)} \end{pmatrix}.$$

At last if $\tilde{A}_{\text{gen}} = \langle X, A, Y, \tilde{r}^{(0)}, \tilde{R} \rangle$ is a partial generalized i-automaton then for $\tilde{A}_{\text{gen}}$ a correction operation is defined as

$$\tilde{A}_{pr} = Cor\ \tilde{A}_{\text{gen}} = \langle X, A, Y, \tilde{p}^{(0)}, R \rangle,$$

where $\tilde{p}^{(0)} = Cor\ \tilde{r}^{(0)}$, $\tilde{P} = Cor\ \tilde{R}$. This correction operation is important for many areas of the partial pi-automata theory, in particular for the minimization of such automata [2].

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