ON FUNCTIONS HAVING THE SAME INTEGRAL ON CONGRUENT SEMIDISKS

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In [1] Proizvolov proved the following statement. If a real valued bounded continuous function defined in the Euclidean 2-space has the same constant integral on every unit square, then the function itself is constant. Later Maljugin [2] succeeded in proving the same, without the restriction of boundedness.

The following problem seems to be interesting. What kind of figures can replace the square in the statement mentioned above? Already in [1] we can find a simple example which shows that the statement is not true for disks of unit radius. This is not so surprising because the disks are invariant under rotation while the squares are not.

Our main result is the following. If a real valued continuous function defined in the Euclidean 2-space has the same constant integral on every semidisk of unit radius, then the function itself is constant.

Throughout the paper $\mathbf{N}$, $\mathbf{R}^n$, λ^n denote the set of integers, the n-dimensional Euclidean space and the Lebesgue-measure on $\mathbf{R}^n$ respectively.

In the paper an important role is played by the following functional equation

$$(1)\qquad F(x+u)+F(x-u) = F(x+v)+F(x-v)$$

where $F:\mathbf{R}^n\to\mathbf{R}$ is the unknown function and x, u, $v\in\mathbf{R}^n$, $|u| = |v| = 1$. First we prove two lemmas concerning equation (1), which will be used in the proof of our main theorem.

LEMMA 1. Suppose that the continuous function $F:\mathbf{R}^n\to\mathbf{R}$ satisfies the equation (1). Then there exist continuous functions $f_i:\mathbf{R}\to\mathbf{R}$ $(i = 1, 2,\dots, n)$ so that

$$(2)\qquad F(x) = \sum_{i=1}^{n} f_i(x_i) \qquad x = (x_1, x_2, \dots, x_n)\in\mathbf{R}^n.$$

PROOF. For $n = 1$ the statement is obvious. Now suppose that the statement is true for n and the continuous function $F:\mathbf{R}^{n+1}\to\mathbf{R}$ satisfies the equation (1). Then the function $F^*:\mathbf{R}^n\to\mathbf{R}$ defined by

$$F^*(x) = F(x, 0) \quad x\in\mathbf{R}^n$$

has also the property (1), and by our assumption, it has the form

$$F^*(x) = \sum_{i=1}^{n} f_i(x_i) \quad x = (x_1, x_2, \dots, x_n)\in\mathbf{R}^n. \tag{3}$$

For any $0\neq x\in\mathbf{R}^n$ let the functions $\Delta_x:\mathbf{R}\to\mathbf{R}$ and ${}^{(k)}\Delta_x^j:\mathbf{R}\to\mathbf{R}$ ($k\in\mathbf{N}$, $k>|x|/2$, $j = 1, 2, \dots, k$) be defined as follows:

$$\begin{aligned} \Delta_x(t) &= F(x, t)-F(0, t) \\ {}^{(k)}\Delta_x^j(t) &= F\left(\frac{j}{k}x, t\right)-F\left(\frac{j-1}{k}x, t\right). \end{aligned} \qquad t\in\mathbf{R} \tag{4}$$

Obviously $\Delta_x = \sum_{j=1}^{k} {}^{(k)}\Delta_x^j$ for all k. Furthermore we show, using the notation $\alpha = \arcsin |x|/2k$, that the functions ${}^{(k)}\Delta_x^j$ are periodic with the period $2p_x^k = 2\cos\alpha$ ($j = 1, 2, \dots, k$). Substituting the vectors

$$\left(\frac{2j-1}{2k}x, t+p_x^k\right), \quad \left(\frac{1}{2k}x, p_x^k\right), \quad \left(\frac{1}{2k}x, -p_x^k\right)$$

into (1) for x, u, v respectively we have

$$\begin{aligned} {}^{(k)}\Delta_x^j(t+2p_x^k) &= F\left(\frac{j}{k}x, t+2p_x^k\right)-F\left(\frac{j-1}{k}x, t+2p_x^k\right) = \\ &= F\left(\frac{j}{k}x, t\right)-F\left(\frac{j-1}{k}x, t\right) = {}^{(k)}\Delta_x^j(t). \end{aligned}$$

Thus Δ_x is periodic with the period $2p_x^k$ for all k, hence $q_x^k = 2(p_x^{k+1}-p_x^k)$ is also a period. Since $\lim_{k\to\infty} q_x^k=0$, the function Δ_x is constant on a dense subset of $\mathbf{R}$ and by the continuity it is constant on the whole $\mathbf{R}$:

$$\Delta_x(t) = \Delta_x(0) = F^*(x)-F(0, 0).$$

Substituting this into (4) we get the desired result:

$$F(x, t) = F^*(x)+F(0, t)-F(0, 0) = \sum_{i=1}^{n} f_i(x_i)+f_{n+1}(t)$$

where $\qquad f_{n+1}:\mathbf{R}\to\mathbf{R}, \quad f_{n+1}(t) = F(0, t)-F(0, 0).$

LEMMA 2. Let $H_1\subset\mathbf{R}^n (n\geq 2)$ be a Lebesgue-measurable set with finit positive measure and suppose that

$$(x_1, x_2, \dots, x_n)\in H_1 \text{ implies } (-x_1, x_2, \dots, x_n)\in H_1 \tag{5}$$

Furthermore let $F: \mathbf{R}^n \to \mathbf{R}$ be a continuous solution of the functional equation (1) such that

$$\int_{T(H_1)} F(x)\, d\lambda^n(x) = 0 \tag{6}$$

for all motions $T: \mathbf{R}^n \to \mathbf{R}^n$. Then F is identically zero.

PROOF. By Lemma 1. there are functions $f_i: \mathbf{R} \to \mathbf{R}$ such that

$$F(x) = \sum_{i=1}^{n} f_i(x_i) \qquad x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n .$$

Without loss of generality we can assume that $F(0) = 0$ and $f_i(0) = 0$ for $i = 1, 2, \ldots, n$.

First we show that f_1 is identically zero. For any $t \in \mathbf{R}$ let $T_t: \mathbf{R}^n \to \mathbf{R}^n$ be the translation

$$T_t(x_1, x_2, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n) .$$

Then by (6)

$$\sum_{i=1}^{n} \int_{H_1} F_i(x)\, d\lambda^n(x) = \int_{H_1} F(x)\, d\lambda^n(x) = \tag{7}$$

$$= \int_{T_t(H_1)} F(x)\, d\lambda^n(x) = \sum_{i=1}^{n} \int_{T_t(H_1)} F_i(x)\, d\lambda^n(x) ,$$

where

$$F_i: \mathbf{R}^n \to \mathbf{R}, \quad F_i(x) = f_i(x_i) \quad (i = 1, 2, \ldots, n) .$$

For $i = 2, 3, \ldots, n$ we have

$$\int_{H_1} F_i(x)\, d\lambda^n(x) = \int_{T_t(H_1)} F_i(x)\, d\lambda^n(x) .$$

Thus (7) implies that

$$\int_{H_1} F_1(x)\, d\lambda^n(x) = \int_{T_t(H_1)} F_1(x)\, d\lambda^n(x) . \tag{8}$$

Now let $M: D\ (\subset \mathbf{R}) \to \mathbf{R}$ denote the Lebesgue-measurable function defined almost everywhere by the formula

$$M(s) = \lambda^{n-1}(H_1 \cap \{x \in \mathbf{R}^n \mid x_1 = s\}) \quad s \in D .$$

Then by the Fubini Theorem and (8) we come to

$$\int_{-\infty}^{\infty} f_1(s)\, M(s)\, d\lambda(s) = \int_{H_1} F_1(x)\, d\lambda^n(x) = \tag{9}$$

$$= \int_{T_t(H_1)} F_1(x)\, d\lambda^n(x) = \int_{-\infty}^{\infty} f_1(s)\, M(s-t)\, d\lambda(s).$$

By (5) M is an even function, thus (9) implies

$$(10)\quad \int_0^\infty [f_1(s)+f_1(-s)]\, M(s)\, d\,\lambda(s) = \int_0^\infty [f_1(t+s)+f_1(t-s)]\, M(s)\, d\,\lambda(s)\,.$$

On the other hand — as it will be shown later — we have

$$(11)\qquad f_1(t+s)+f_1(t-s)-2f_1(t) = f_1(s)+f_1(-s) \quad s\in\mathbf{R}$$

which together with (10) proves the statement:

$$f_1(t)\cdot\lambda^n(H_1) = 2\cdot f_1(t)\int_0^\infty M(s)\, d\,\lambda(s) = 0\,.$$

It can be proved in the same way that the other functions f_i are also identically zero (instead of H_1 one should use in the proof a set H_i which can be obtained from H_1 by a suitable rotation).

The equation (11) for $|s|<1$ is a direct consequence of the functional equation (1); letting $\alpha = \arcsin |s|$ and $r = \cos\alpha$, we have

$$(12)\quad [f_1(s)+f_2(r)]+[f_1(-s)+f_2(-r)] = f_2(1)+f_2(-1) = [f_1(t)+f_2(1)]+$$
$$+[f_1(t)+f_2(-1)]-2f_1(t) = [f_1(t+s)+f_2(r)]+[f_1(t-s)+f_2(-r)]-2f_1(t)\,.$$

Now suppose that (11) holds for $|s|<m\in\mathbf{N}$. Let $m\le s<m+1$ then using (1) and our assumption we get

$$f_1(t+s)+f_1(t-s)-2f_1(t) = [2f_1(t+s-1)+f_2(1)+f_2(-1)-f_1(t+s-2)]+$$
$$+[2f_1(t-s+1)+f_2(1)+f_2(-1)-f_1(t-s+2)]-2f_1(t) = 2[f_2(1)+f_2(-1)]+$$
$$+2[f_1(s-1)+f_1(-s+1)+2f_1(t)]-[f_1(s-2)+f_1(-s+2)+2f_1(t)]-2f_1(t) =$$
$$=2[f_1(s-1)+f_1(-s+1)]-[f_1(s-2)+f_1(-s+2)]+2[f_2(1)+f_2(-1)]\,.$$

Thus $f_1(t+s)+f_1(t-s)-2f_1(t)$ does not depend on t and so, choosing $t = 0$; for $m\le s<m+1$ we have (11). Finally (12) and the last argument prove (11) by induction.

THEOREM 1. Suppose that the continuous function $F:\mathbf{R}^2\to\mathbf{R}$ has the same constant integral on the images $T(K_0)$ of the semidisk

$$K_0 = \{(x,\, y)\in\mathbf{R}^2 \,|\, x^2+y^2\le 1\,, \quad y\ge 0\}$$

for all motions $T:\mathbf{R}^2\to\mathbf{R}^2$, i.e.

$$\int_{T(K_0)} F(x,\, y)\, d\,\lambda^2(x,\, y) = c\,.$$

Then F is identically constant.

PROOF. According to the preceeding lemmas it is sufficient to show that F is a solution of the functional equation (1).

For any $\alpha \in \mathbf{R}$, $t \in \mathbf{R}$ let $T_t^\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be the translation

$$T_t^\alpha(x, y) = (x + t \cdot \cos\alpha, y + t \sin\alpha) \quad (x, y) \in \mathbf{R}^2 .$$

Let the semidisk K_α be defined by turning K_0 around the origin by the angle α. Then the function $I_\alpha : \mathbf{R} \to \mathbf{R}$ defined by

$$I_\alpha(t) = \int_{T_t^\alpha(K_\alpha)} F(x, y)\, d\lambda^2(x, y) = c \quad t \in \mathbf{R}$$

is differentiable at $t = 0$, and

$$I'_\alpha(0) = \int_\alpha^{\alpha+\pi} F(\cos\varphi, \sin\varphi)\cos(\varphi - \alpha)\, d\varphi = 0 . \tag{13}$$

It is easy to see that the function $p : \mathbf{R} \to \mathbf{R}$ defined by

$$p(\varphi) = F(\cos\varphi, \sin\varphi) \quad \varphi \in \mathbf{R}$$

is periodic with the period 2π, continuous and for any $\alpha \in \mathbf{R}$

$$\int_0^\pi p(\varphi + \alpha) \cdot \cos\varphi\, d\varphi = \int_\alpha^{\alpha+\pi} p(\varphi) \cdot \cos(\varphi - \alpha)\, d\varphi = 0 . \tag{14}$$

Denote by S_n the n-th partial sum of the Fourier-series of p, i.e.

$$S_n(\varphi) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos k\varphi + b_k \sin k\varphi)$$

which converges to p in L_2 norm:

$$\|S_n - p\|^2 = \int_0^{2\pi} (S_n(\varphi) - p(\varphi))^2\, d\varphi \to 0 .$$

Then the Cauchy-Schwarz inequality yields

$$\left| \int_0^\pi S_n(\varphi + \alpha) \cdot \cos\varphi\, d\varphi \right| = \left| \int_0^\pi [S_n(\varphi + \alpha) - p(\varphi + \alpha)] \cdot \cos\varphi\, d\varphi \right| \leq$$

$$\leq \sqrt{\int_0^\pi [S_n(\varphi + \alpha) - p(\varphi + \alpha)]^2\, d\varphi} \cdot \sqrt{\int_0^\pi \cos^2\varphi\, d\varphi} \leq$$

$$\leq \sqrt{\int_0^{2\pi} [S_n(\varphi + \alpha) - p(\varphi + \alpha)]^2\, d\varphi} \cdot \sqrt{\frac{\pi}{2}} = \|S_n - p\| \sqrt{\frac{\pi}{2}} \to 0 .$$

Now, integrating term by term, we have

$$\int_0^\pi S_n(\varphi+\alpha)\cos\varphi\, d\varphi = \frac{a_1\pi}{2}\cos\alpha+\frac{b_1\pi}{2}\sin\alpha +$$

$$+\sum_{k=1}^{\left[\frac{n}{2}\right]}\left(\frac{4kb_{2k}}{4k^2-1}\cos 2k\alpha-\frac{4ka_{2k}}{4k^2-1}\sin 2k\alpha\right)\quad \alpha\in\mathbf{R},$$

i.e. by $n\to\infty$ we obtain

$$\frac{a_1\pi}{2}\cos\alpha+\frac{b_1\pi}{2}\sin\alpha+\sum_{k=1}^{\infty}\left(\frac{4kb_{2k}}{4k^2-1}\cos 2k\alpha-\frac{4ka_{2k}}{4k^2-1}\sin 2k\alpha\right)=0. \tag{15}$$

Next we show that the trigonometric series (15) are uniformly convergent. Indeed

$$\sum_{k=1}^{\infty}\left(\left|\frac{4kb_{2k}}{4k^2-1}\cos 2k\alpha\right|+\left|\frac{4ka_{2k}}{4k^2-1}\sin 2k\alpha\right|\right)\le\sum_{k=1}^{\infty}\left(\frac{4k|b_{2k}|}{4k^2-1}+\frac{4k|a_{2k}|}{4k^2-1}\right) \tag{16}$$

and according to the Bessel-inequality we have

$$\sum_{k=1}^{\infty}(b_{2k}^2+a_{2k}^2)\le\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)\le\frac{1}{\pi}\int_0^{2\pi}p^2(\varphi)\,d\varphi<\infty \tag{17}$$

and on the other hand

$$2\sum_{k=1}^{\infty}\left(\frac{4k}{4k^2-1}\right)^2\le 2\sum_{k=1}^{\infty}\left(\frac{4k}{2k^2}\right)^2=8\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty. \tag{18}$$

(17), (18) and the Cauchy-Schwarz inequality yields

$$\sum_{k=1}^{\infty}\left(\frac{4k}{4k^2-1}|b_{2k}|+\frac{4k}{4k^2-1}|a_{2k}|\right)\le\sqrt{2\sum_{k=1}^{\infty}\left(\frac{4k}{4k^2-1}\right)^2}\sqrt{\sum_{k=1}^{\infty}(b_{2k}^2+a_{2k}^2)}<\infty$$

which together with (16) proves the uniform convergence of the series (15). Hence (15) is the Fourier-series of the identically zero function, and so

$$a_1=0,\quad b_1=0,\quad a_{2k}=0,\quad b_{2k}=0\quad (k\in\mathbf{N}).$$

Now let us notice that for any $\varphi\in\mathbf{R}$ and $k\in\mathbf{N}$

$$\cos(2k+1)(\varphi+\pi)+\cos(2k+1)\varphi=0,$$
$$\sin(2k+1)(\varphi+\pi)+\sin(2k+1)\varphi=0.$$

Thus for any $\varphi\in\mathbf{R}$ and $n\in\mathbf{N}$ it follows

$$S_n(\varphi)+S_n(\varphi+\pi)=a_0+\sum_{k=1}^{\left[\frac{n-1}{2}\right]}a_{2k+1}[\cos(2k+1)\varphi+\cos(2k+1)(\varphi+\pi)]+$$

$$+\sum_{k=1}^{\left[\frac{n-1}{2}\right]}b_{2k+1}[\sin(2k+1)\varphi+\sin(2k+1)(\varphi+\pi)]=a_0.$$

Denoting by $\tau_\pi : \mathbf{R} \to \mathbf{R}$ the translation $\tau_\pi(\varphi) = \varphi + \pi$, we can see that $S_n + S_n \circ \tau_\pi = a_0$ converges to $p + p \circ \tau_\pi$ in the L_2 norm. Thus

$$p(\varphi) + p(\varphi + \pi) = a_0 \tag{19}$$

holds for a.e. $\varphi \in \mathbf{R}$, and because of the continuity of (19) holds everywhere.

Of course the procedure described above can be carried out for any unit disk of $\mathbf{R}^2$, and leads to a result analogous to (19). Namely this means for the function F that for any $(x, y) \in \mathbf{R}^2$ and $-\pi \le \alpha \le \pi$ we have

$$F(x + \cos\alpha, y + \sin\alpha) + F(x - \cos\alpha, y - \sin\alpha) = a_0(x, y)$$

where the function $a_0 : \mathbf{R}^2 \to \mathbf{R}$ does not depend on α. This last equality is exactly the functional equation (1) in the two dimensional case.

REFERENCES

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