ON APPROXIMATION OF HYDROLOGICAL FUNCTIONS

by

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1. Introduction

In their paper [1] L. Góczán and A. F. Szász introduced some hydrological functions of new type to investigate the agronomical and technical economy of water-supplies.

In many cases they had to determine a good approximating polynomial for the hydrological functions $f(x)$ under the following conditions:

(a) $f'(x)$ is continuous (say) in $[-1,1]$ (i.e., $f' \in C$).

(b) One can measure the data $f(1)$, $f'(1)$ and

$$f(x_k) \quad (k = 1,2, \ldots, n, \ -1 \leqq x_k < 1, \ x_k \neq x_j \text{ if } k \neq j.).$$

As an easy calculation shows the uniquely determined polynomial $G_n(f, x)$ of degree $\leqq n+1$ satisfying the conditions
(a) and (b) has the form

$$G_n(f, x) = \sum_{k=1}^{n} f(x_k)\, l_k(x) \frac{(x-1)^2}{(x_k-1)^2} + f(1) \frac{\Omega(x)}{\Omega(1)} \left[1 - \frac{\Omega'(1)}{\Omega(1)}(x-1)\right] +$$

$$+ f'(1) \frac{\Omega(x)}{\Omega(1)}(x-1) \tag{1.1}$$

where

$$\begin{cases} \Omega(x) = \Omega_n(x) = c_n \prod_{k=1}^{n} (x - x_k) \quad (c_n \neq 0), \\ l_k(x) = l_{kn}(x) = \dfrac{\Omega(x)}{\Omega'(x_k)(x-x_k)} \quad (k = 1, \ldots, n). \end{cases} \tag{1.2}$$

A very natural question (which was raised in [1], too) is how to choose the nodes $x_k = x_{kn}$ ($k = 1,2, \ldots, n$; $n = 1,2, \ldots$) to ensure that

$$\lim_{n \to \infty} \|G_n(f, x) - f(x)\| = 0 \quad \text{whenever} \quad f' \in C. \tag{1.3}$$

Here, as usual, $\|g\| = \max_{-1 \leqq x \leqq 1} |g(x)|$ for $g \in C$.

2. Results

2.1. It is easy to see that using the equidistant nodes $x_{kn} = -1+k(n+1)^{-1}$ (which were applied in [1]), the "Lebesgue constant" of the process (1.1)

$$\lambda_n = \|\lambda_n(x)\| \overset{\text{def}}{=} \left\| \sum_{k=1}^{n} |l_k(x)| \frac{(x-1)^2}{(x_k-1)^2} + \right.$$

$$\left. + \frac{|\Omega(x)|}{|\Omega(1)|} \left| 1 - \frac{\Omega'(1)}{\Omega(1)}(x-1) \right| + \left| \frac{\Omega(x)}{\Omega(1)}(x-1) \right| \right\|$$

very rapidly tends to infinity with n, from where we shall obtain, that *for the equidistant nodes* (1.3) *generally does not hold* (see 3.6).

But considering the Jacobi nodes, i.e. the roots

$$(2.1) \qquad -1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{1n}^{(\alpha,\beta)} < 1 \quad (\alpha, \beta > -1)$$

of the polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n defined by

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx} \right)^n [(1-x)^{n+\alpha} (1+x)^{n+\beta}]$$

we can state as follows

THEOREM 2.1. *If* $3 < \alpha = \beta+4 \leqq 4.5$, *for the* $G_n(\alpha, \beta)(f, x) = G_n(f, x)$ *process defined on the nodes* (2.1) *the relation* (1.3) *is valid. More exactly, if* $\omega(f', t)$ *denotes the modulus of continuity of* $f'(x) \in C$, *then*

$$(2.2) \qquad \|G_n(\alpha, \beta)(f, x) - f(x)\| = \begin{cases} O\left(\dfrac{\ln n}{n}\right) \omega\left(f', \dfrac{1}{n}\right) & \text{if } 3 < \alpha \leqq 3.5, \\ O(n^{\alpha-4.5})\, \omega\left(f', \dfrac{1}{n}\right) & \text{if } \alpha > 3.5. \end{cases}$$

2.2. Here we state, the Theorem 2.1, in certain sense, is not far from the best possible one. This can be formulated as follows.

THEOREM 2.2. *If* $\alpha = \beta+4$ *and* $\alpha > 4{,}5$, *then one can construct a function* $f_1(x)$ *such that* $f_1'(x) \in C$ *and*

$$(2.3) \qquad \overline{\lim_{n\to\infty}} \|G_n(\alpha, \beta)(f_1, x) - f_1(x)\| = \infty.$$

2.3. Similar results can be obtained for $\beta = 1$. We omit the details.

3. Proofs

3.1. We shall use the following relations (sometimes omitting the superfluous notations).

$$(3.1) \qquad P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x).$$

if $$x_{kn} = x_{kn}^{(\alpha,\beta)} = \cos \vartheta_{kn}^{(\alpha,\beta)}, \quad x_{0n} = 1, \quad x_{n+1,n} = -1,$$

(3.2) $$\vartheta_{k+1,n}^{(\alpha,\beta)} - \vartheta_{kn}^{(\alpha,\beta)} \sim \frac{1}{n} \quad (k = 0, 1, \ldots, n)$$

moreover, if $x_{j(n),n}$ is the nearest root to $x\,(1 \leqq j \leqq n)$ then

(3.3) $$|x - x_{kn}| \sim \frac{|j^2 - k^2|}{n^2} \quad (k \neq j;\ k = 0, 1, 2, \ldots, n+1),$$

(3.4) $$|P_n'^{(\alpha,\beta)}(x_{kn})| \sim k^{-\alpha-3/2} n^{\alpha+2} \quad (0 < \vartheta_k \leqq \pi - \varepsilon),$$

(3.5) $$|P_n^{(\alpha,\beta)}(x)| \sim |x - x_j|\, \vartheta_j^{-\alpha-3/2}\, n^{1/2} \sim |\vartheta - \vartheta_j|\, \vartheta_j^{-\alpha-1/2} n^{1/2}$$

uniformly for $\vartheta \in [0, \pi - \varepsilon]$.

At these formulae $x = \cos \vartheta$, the „$\sim$" depends on α and β (see [2], (1.1), (4.1.3), (8.9.2) and [3], [4]).

Denoting by $Q_n(x) = Q_n(f, x)$ the polynomial of degree $\leqq n$ for which

(3.6) $$|f^{(i)}(x) - Q_n^{(i)}(x)| = 0(1)\left(\frac{\sin \vartheta}{n}\right)^{1-i} \omega\left(f', \frac{\sin \vartheta}{n}\right) \quad (i = 0, 1)$$

(see [5]) we can write by (1.1)

(3.7) $$G_n(f, x) - f(x) = G_n(f, x) - Q_n(f, x) + Q_n(f, x) - f(x) =$$
$$= G_n(f - Q_n, x) + Q_n(f, x) - f(x) =$$
$$= \sum_{k=1}^{n} [f(x_k) - Q_n(x_k)]\, l_k(x)\left(\frac{x-1}{x_k - 1}\right)^2 + 0(1)\frac{\sin \vartheta}{n}\, \omega\left(f', \frac{\sin \vartheta}{n}\right).$$

3.2. For the sum, using the formulae (3.1)–(3,6), we can write if $x \geqq 0$

$$\sum_{k=1}^{n} \cdots = 0(1)\left[\sum_{k=1}^{n}{}' \frac{\sin \vartheta_k}{n}\, \omega\left(\frac{\sin \vartheta_k}{n}\right)\left|\frac{P_n(x)}{P_n'(x_k)(x - x_k)}\right|\left(\frac{x-1}{x_k - 1}\right)^2 + \right.$$
$$\left. + \frac{\sin \vartheta_j}{n}\, \omega\left(\frac{\sin \vartheta_j}{n}\right) \frac{\vartheta_j^{-\alpha-3/2} n^{1/2}}{j^{-\alpha-3/2} n^{\alpha+2}}\right] = 0(1) \sum_{k=1}^{\left[\frac{3n}{4}\right]}{}' \frac{\vartheta_k}{n}\, \omega\left(\frac{\vartheta_k}{n}\right) \cdot$$
$$\cdot \frac{\vartheta_j^{-\alpha-1/2} n^{-1/2} n^2}{k^{-\alpha-3/2} n^{\alpha+2} |k+j|\,|k-j|} \frac{j^4 n^{-4}}{k^4 n^{-4}} + 0(1) \sum_{k=1}^{\left[\frac{n}{4}\right]} \frac{\vartheta_k}{n}\, \omega\left(\frac{\vartheta_k}{n}\right) \cdot$$
$$\cdot \frac{\vartheta_j^{-\alpha-1/2} n^{-1/2}}{k^{-\beta-3/2} n^{\beta+2}} \frac{j^4}{n^4} \stackrel{\text{def}}{=} S_1 + S_2$$

where Σ' means that $k \neq j$.

$$S_1 = 0(1)\frac{1}{n^2}\sum_{k=1}^{n}{}'\,\omega\left(\frac{k}{n^2}\right)\frac{j^{3,5-\alpha}\,k^{\alpha-1.5}}{|k+j|\,|k-j|} =$$

$$= 0(1)\left(n^{-2}\sum_{k<\frac{j}{2}}\ldots+n^{-2}\sum_{\frac{j}{2}\leqq k\leqq 2j}{}'\ldots+n^{-2}\sum_{k>2j}\right) \stackrel{\text{def}}{=} 0(1)(I_1+I_2+I_3).$$

Now

$$I_1 = 0(n^{-2})\,j^{1,5-\alpha}\,\omega\left(\frac{1}{n}\right)\sum_{k=1}^{j} k^{\alpha-1.5}$$

so $I_1 = 0(n^{-1})\,\omega\left(\frac{1}{n}\right)$ if $\alpha>3$.

For I_2 we can write

$$I_2 = 0(n^{-2})\,j\omega\left(\frac{1}{n}\right)\sum_{k=j+1}^{2j}(k-j)^{-1} = 0\left(\frac{\ln n}{n}\right)\omega\left(\frac{1}{n}\right).$$

As I_3, one can estimate as follows

$$I_3 = 0(n^{-2})\,j^{3,5-\alpha}\sum_{k=2j}^{n}\omega\left(\frac{k}{n^2}\right)k^{\alpha-3,5}$$

which is $0\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)$ if $3<\alpha\leqq 3,5$ and $0(n^{\alpha-4,5})\,\omega\left(\frac{1}{n}\right)$ if $\alpha>3,5$.

Let us estimate S_2. We have

$$S_2 = 0(1)\frac{j^{3,5-\alpha}}{n^{8-\alpha-\beta}}\sum_{k=1}^{n}\omega\left(\frac{k}{n^2}\right)k^{\beta+2,5} = 0(1)\frac{j^{3,5-\alpha}}{n^4}\sum_{k=1}^{n}\omega\left(\frac{k}{n^2}\right)k^{\alpha-1,5}$$

which can be estimated by $0\left(\frac{1}{n}\right)\omega\left(\frac{1}{n}\right)$ if $3<\alpha\leqq 3,5$ and by $0(n^{\alpha-4,5})$. $+\omega\left(\frac{1}{n}\right)$ if $\alpha>3,5$.

3.3. Let now $x<0$. Then the corresponding expression with $t = n+1-j$ are as follows

$$S_1^* = 0(1)\frac{1}{n^2}\sum_{k=1}^{n}{}'\,\omega\left(\frac{k}{n^2}\right)\frac{t^{-\beta-1/2}\,k^{\beta+2,5}}{|k+t|\,|k-t|},$$

$$S_2^* = 0(1)\frac{t^{-\beta-1/2}}{n^{\alpha-\beta}}\sum_{k=1}^{n}\omega\left(\frac{k}{n^2}\right)k^{\alpha-1,5},$$

i.e. S_1^* and S_2^* has the same form as S_1 and S_2, respectively, if we consider $\alpha = \beta+4$. These estimations give Theorem 2.1.

3.4. To prove Theorem 2.2, we apply the following statement, which is a special case of [6], **Theorem 3.1.**

If for the sequence of linear operators $T_n(f, x)$ $(f \in C)$ and the functions $g_n(x)$ $(n = 1,2, \dots)$ we have

(a1) $g_n(x) \in C$,

(a2) $T_n(g_n, z_n) \geqq c_n\, \mu_n(z_n)$ *for certain* $\{z_n\} \subset [-1,1]$ (*definition of* $\mu_n(z_n)$);

$$(B^*) \qquad \tilde{f}(x) \overset{\text{def}}{=} \sum_{i=1}^{\infty} e_{n_i} g_{n_i}(x) \in C \quad \textit{for any} \quad \{n_i\} \subset \{p_r\}$$

where $\{p_r\}$ *is a certain fixed sequence of indices and* $0 < e_n \leqq e_{n+1} \leqq 1$;

$$(C^*) \qquad c_5\, \mu_n(z_n) > \sum_{i=k+1}^{\infty} e_{n_i} |T_{n_k}(g_{n_i}, z_{n_k})| + \sum_{i=k}^{\infty} e_{n_i} |(g_{n_i}\, z_{n_k})|$$

$(k = 1,2, \dots;\ 0 < C_4 < C_5$ for arbitrary $\{n_i\} \subset \{p_r\}$; *then with a suitable* c_8 *and* $\{s_i\} \subset \{p_r\}$ *with* $f_1(x) = c_8 \tilde{f}(x)$

$$(3.8) \qquad T_n(f_1, z_n) - f_1(z_n) > e_n\, \mu_n(z_n) \qquad (n = s_1, s_2, \dots).$$

(Here c_i are fixed constants.)

3.5. Let $\alpha - 4{,}5 = \varepsilon > 0$, $z_4 = (1 + x_{1n})/2$ and

$$(3.9) \qquad g_n(x_{kn}) = \begin{cases} \dfrac{\sin \vartheta_k}{n}\, \omega_1\left(\dfrac{\sin \vartheta_k}{n}\right) \operatorname{sign} l_k(z_n) \text{ if } -1 < x_k \leqq 0, \\ 0 \text{ if } k = n+1 \text{ or } x_k > 0, \end{cases}$$

where $\omega_1(t)$ is a modulus of continuity.

In the interval (x_{k+1}, x_k) let $g_n(x)$ be the Hermite interpolatory polynomial of degree $\leqq 3$ for which

$$g_n'(x_{kn}) = g_n'(x_{k+1,\,n}) = 0 \qquad (k = 0,1, \dots, n).$$

Let $T_n = G_n$. By (3.9) and (1.1)

$$G_n(g_n, z_n) = \sum_{-1 < x_k \leqq 0} \frac{\sin \vartheta_k}{n}\, \omega_1\left(\frac{\sin \vartheta_k}{n}\right) |l_k(z_n)| \left(\frac{z_n - 1}{x_k - 1}\right)^2 \overset{\text{def}}{=} \mu_n(z_n).$$

By (3.1)–(3.5) we obtain, as at the estimation of S_2

$$(3.10) \quad \mu_n(z_n) \sim \frac{1}{n^4} \sum_{k=1}^{n} \omega_1\left(\frac{k}{n^2}\right) k^{\alpha - 1{,}5} \geqq c\, \omega_1\left(\frac{1}{n^2}\right) n^{\varepsilon} \ (n = 1, 2, \dots)$$

i.e. we can suppose $\mu_n(z_n)\nearrow\infty$ for a „bad" $\omega_1(t)$. Let us see now (B*). By definition it is easy to verify, that $\omega(g_n', t)\sim\omega_1(t)$, i.e. supposing $\sum_{i=1}^{\infty} e_{p_i}<\infty$, we obtain for any $\{n_i\}\subset\{p_r\}$

$$(3.11)\qquad \omega(\tilde{f}', t)\leqq c\sum_{i=1}^{\infty} e_{n_i}\,\omega(g_{n_i}', t) = 0(\omega_1(t)),$$

which is more than (B*).

To prove (C*), we remark, that $|g_n(x)|\leqq 1/n$. So, if we choose the sequence $\{p_i\}$ such that

$$\frac{\lambda_{p_k}}{p_{k+1}}\leqq 1 \qquad (k = 1, 2, \dots)$$

we obtain (C*).

So by (3.8)–(3.11)

$$G_n(f_1, z_n)-f_1(z_n)>e_n\,\omega_1\left(\frac{1}{n^2}\right)n^{\varepsilon} \quad (n = s_1, s_2, \dots)$$

which, with a suitable $\{e_i\}$, is more than (2.3).

3.6. For the equidistant nodes one can use analogous argument, using the fact that for certain $\{k_n\}$ $\|l_{k_n, n}(x)\|>(1{,}5)^{n/2}$ $(n\geqq n_0)$ (see e.g. [7] Part 3, Chapter 2, § 3). We omit the further details.

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