THE L-FUZZY KLEENE THEOREM

By

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1. Introduction

Identifying the subsets of a set U with their characteristic functions we can generalize the notion of a subset. A fuzzy subset of U is a mapping from U into the closed interval [0, 1] (Zadeh [1]). The notion of a fuzzy event has been introduced by Thiele [2]. The fuzzy events over a finite alphabet X are the fuzzy subsets of the free semigroup X^*. Goguen [3] recognizes that the ordering on the real numbers is the essential point of the generalization. Therefore he defines the so-called L-fuzzy sets as mappings from a set into an arbitrary lattice.

In the present paper we deal with the L-fuzzy generalization of events, where L is a distributive lattice with zero and unit elements, and we define the notion of L-fuzzy automata. We shall show, that L-fuzzy events are represented by L-fuzzy automata if and only if they are L-regular. Wechler and Dimitrov studied a more general notion ,the R-fuzzy events ([4]), where R is an ordered semiring. They proved a similar theorem, but their iteration concept is weaker, it is defined only for ε-free R-fuzzy events. In our approach, the regular operations are total, and this makes an essential difference between the proofs. Our proof is constructive, and analogous to the proof of the classical Kleene theorem.

2. L-fuzzy events, L-regularity, L-fuzzy automata

L-fuzzy events are defined by means of a distributive lattice L with zero and unit elements. The operations are denoted by $\vee$ and $\wedge$.

Definition 1. An L-fuzzy event over a finite alphabet X is a mapping from X^* into L where L is a distributive lattice with zero and unit elements. Notation: Let $L(X)$ denote the set of all L-fuzzy events over X. The usual events and words of alphabet X are considered to be elements of $L(X)$ while they can be identified with their characteristic functions.

The following operations are introduced on $L(X)$.

Definition 2. Let $g,\ h \in L(X)$, $l \in L$. Then the scalar product $lg = gl$, the union $g \cup h$, the concatenation gh and the iteration g^* is defined for all $w \in X^*$ by

$$1. \quad (lg)(w) = (gl)(w) = l \wedge g(w)$$

$$2. \quad (g \cup h)(w) = g(w) \vee h(w)$$

$$3. \quad gh(w) = \bigvee_{\substack{u, v \in X^* \\ uv = w}} g(u)\, h(v)$$

$$4) \quad g^*(w) = \begin{cases} \bigvee\limits_{\substack{w_1 \dots w_n \in X^+ \\ w_1, \dots, w_n = w}} \bigwedge\limits_{k=1}^{n} g(w_k) & \text{if } w \in X^+ \\ 1 & \text{if } w \notin X^+ \end{cases}$$

where $X^+ = X^* - \{\varepsilon\}$ and ε is the empty word.

The L-fuzzy operations union, concatenation and iteration have the same properties as in the classical theory. (The union is commutative, associative, the production is associative and distributive to the union. For the iteration

$$g^* = \bigcup_{i=0}^{\infty} g^i \text{ and } g^* = (g\, g^*) \cup \varepsilon$$

hold.

For the scalar product the following equations hold:

$$l(g \cup h) = lg \cup lh$$

$$l(gh) = (lg)h = g(lh)$$

$$(lg)^* = lg^* \cup \varepsilon.$$

Notation. Let $L(X)_{\text{reg}}$ denote the smallest subset of L-fuzzy events containing the empty word and the letters of X – wich are called elementary events – and closed under the operations (1)–(4) given in Definition 2.

Definition 3. The elements of $L(X)_{\text{reg}}$ are called L-regular events.

Another equivalent definition for L-regularity can be given with the help of expressions. We now define L-fuzzy automata, as direct generalizations of nondeterministic automata. When describing the structure of L-fuzzy automata, the matrices over L will play an important role.

Notation. Let $L^{n,m}$ denote the set of $n \times m$ type matrices over L.

Definition 4. For matrices $A,\ B \in L^{n,m}$, $C \in L^{m,r}$ and $l \in L$, the scalar product lA, the sum $A + B \in L^{n,m}$ and the product $AC \in L^{n,r}$ are defined by

$$(lA)(i,j) = l \wedge A(i,j)$$

$$(A+B)(i,j) = A(i,j) \vee B(i,j)$$

$$(AC)(i,j) = \bigvee_{k=1}^{m} A(i,k) \wedge C(k,j).$$

These operations have the usual properties. Identifying the elements of L with elements of $L^{1,1}$ we get the following correspondence between operations in L and $L^{1,1}$

$\wedge \rightarrow$ matrix product

$\vee \rightarrow$ matrix sum.

Definition 5. An L-fuzzy automaton is a 5-tuple

$$A = \langle S, X, P(x)_{x \in X}, \mu, \eta \rangle \text{ where}$$

$S = \{s^1, s^2, \ldots, s^n\}$: finite nonepty set of states
$X = \{x^1, x^2, \ldots, x^k\}$: finite alphabet, the set of input signals.

$P(x) \in L^{n,n}$ – the transition matrix for the input x.
$\mu \in L^{1,n}$ – the initial vector
$\eta \in L^{n,1}$ – the final vector.

We extend the function $x \rightarrow P(x)$ to the free semigroup X^* by

$$P(\varepsilon)(i,j) = \delta_{ij} \; (\delta_{ij} \text{ is the Kronecker symbol, underestood in } L)$$

$$P(w) = P(x_1)\, P(x_2) \ldots P(x_e) \text{ for } w = x_1 x_2 \ldots x_e.$$

Definition 6. An L-fuzzy event g is said to be representable by L-fuzzy automata if there is an L-fuzzy automaton A with input set X, such that

$$g(w) = \mu P(w)\, \eta \qquad \text{for all } w \in X^*.$$

We denote the function $\mu P(w) \eta$ by $\chi_A(w)$, and we say that χ_A is the L-fuzzy event, accepted by A. So g is called representable by L-fuzzy automaton, if $g = \chi_A$ for some L-fuzzy automaton A.

Notation. Let $L(X)_{\text{aut}}$ denote the set of those L-fuzzy events representable by L-fuzzy automata.

Theorem (Kleene). An L-fuzzy event g is representable by L-fuzzy automata iff g is L-regular.

The proof of the theorem consists of two parts.

a) to show that the L-regular events, are representable by L-fuzzy automata (synthesis of L-fuzzy automata)

b) to show that the L-fuzzy events, representable by L-fuzzy automata are L-regular (analysis of L-fuzzy automata).

3. Synthesis of L-fuzzy automata

We will show that

a) the elementary events are elements of $L(X)_{\text{aut}}$ and
b) $L(X)_{\text{aut}}$ is closed under the operations given in Definition 2.

Lemma 1. Each elementary event is representable by an L-fuzzy automaton.

PROOF. The L-fuzzy automata are direct generalizations of nondeterministc automata, for which this lemma is already proved.

Lemma 2. $L(X)_{\text{aut}}$ is closed under scalar product.

PROOF. Let $l\in L$ and $g\in L(X)_{\text{aut}}$. There exists an L-fuzzy automaton $A=\langle S, X, P(x)_{x\in X}, \mu, \eta\rangle$ for which $g=\chi_A$.
For the L-fuzzy automaton $A_l=\langle S, X, P(x)_{x\in X}, l\mu, \eta\rangle$

$$\chi_{A_l}(w)=(l\mu)P(w)\eta=l(\mu P(w)\eta)=l\chi_A(w)=lg(w),$$

hence A_l represents lg.

Lemma 3. $L(X)_{\text{aut}}$ is closed under union.

PROOF. Let $g_1, g_2\in L(X)_{\text{aut}}$. There exist L-fuzzy automata $A_i=\langle S_i, X, P_i(x)_{x\in X}, \mu_i, \eta_i\rangle$ $i=1, 2$ representing g_i. It can be supposed that $S_1\cap S_2=\emptyset$. The direct sum $A_1\oplus A_2$ is defined by

$$A_1\oplus A_2=\langle S_1\cup S_2, X, P(x), \mu, \eta\rangle \text{ where}$$

$$P(x)=\begin{array}{c} \\ S_1 \\ S_2 \end{array}\begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \left[\begin{array}{c|c} P_1(x) & 0 \\ \hline 0 & P_2(x) \end{array}\right] \end{array}$$

$$\mu=\overset{\begin{array}{cc} S_1 & S_2 \end{array}}{\left[\begin{array}{c|c}\mu_1 & \mu_2\end{array}\right]}, \quad \eta^T=\overset{\begin{array}{cc} S_1 & S_2 \end{array}}{\left[\begin{array}{c|c}\eta_1^T & \eta_2^T\end{array}\right]} \quad \text{in block form.}$$

It is clear that for each $w\in X^*$

$$P(w)=\left[\begin{array}{c|c} P_1(w) & 0 \\ \hline 0 & P_2(w) \end{array}\right]$$

$$\chi_{A_1\oplus A_2}(w)=\left[\begin{array}{c|c}\mu_1 & \mu_2\end{array}\right]\left[\begin{array}{c|c} P_1(w) & 0 \\ \hline 0 & P_2(w) \end{array}\right]\left[\begin{array}{c}\eta_1 \\ \hline \eta_2\end{array}\right]=\mu_1P_1(w)\eta_1\vee\mu_2P_2(w)\eta_2=$$

$$=\chi_{A_1}(w)\vee\chi_{A_2}(w)=g_1(w)\vee g_2(w),$$

therefore $A_1\oplus A_2$ represents $g_1\cup g_2$.

Lemma 4. $L(X)_{\mathrm{aut}}$ is closed under concatenation.

PROOF. Let $g_1, g_2 \in L(X)_{\mathrm{aut}}$. There exist L-fuzzy automata $A_i = \langle S_i, X, P_i(x)_{x\in X}, \mu_i, \eta_i\rangle$ $i = 1, 2$ representing g. It can be supposed, that $S_1 \cap S_2 = \emptyset$. We construct an L-fuzzy automaton A representing $g_1 g_2$.

$$A = \langle S_1 \cup \{s\} \cup S_2, X, P(x)_{x\in X}, \mu, \eta\rangle$$

where $s \notin S_1 \cup S_2$ is a new symbol,

$$P(x) =[1] \begin{array}{c} \\ S_1 \\ s \\ S_2 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & s & S_2 \end{array} \\ \left[\begin{array}{c:c:c} P_1(x) & P_1(x)\,\eta_1 & 0 \\ \hdashline 0 & 0 & \mu_2 P_2(x) \\ \hdashline 0 & 0 & P_2(x) \end{array}\right] \end{array} =$$

$$\mu = \begin{array}{c} \begin{array}{ccc} S_1 & s & S_2 \end{array} \\ \left[\begin{array}{c:c:c} \mu_1 & \mu_1\ \eta_1 & 0 \end{array}\right] \end{array} \quad \text{and}$$

$$\eta = \begin{array}{c} \begin{array}{ccc} S_1 & s & S_2 \end{array} \\ \left[\begin{array}{c:c:c} 0 & \mu_1\ \eta_1 & \mu_2^T \end{array}\right] \end{array} \quad \text{in block form.}$$

It can be proved by very simple induction, that the block form of $P(w)$ $(w = x_1 x_2 \ldots, x_j \in X^*\ j \geq 2)$ is

$$P(w) = \left[\begin{array}{c:c:c} P_1(w) & P_1(w)\,\eta_1 & \bigvee_{l=1}^{j} P(x_1, \ldots, x_l)\,\eta_1\,\mu_2\,P_2(x_{l+1} \ldots x_j) \\ \hdashline 0 & 0 & \mu_2 P_2(w) \\ \hdashline 0 & 0 & P_2(w) \end{array}\right].$$

Let us calculate the values of χ_A for each $w \in X^*$.

a) If $w = \varepsilon$ then

$$\chi_A(\varepsilon) = \left[\begin{array}{c:c:c} \mu_1 & \mu_1\ \eta_1 & 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hdashline \mu_2\ \eta_2 \\ \hdashline \eta_2 \end{array}\right] = \mu_1\,\eta_1 \wedge \mu_2\,\eta_2 = g_1(\varepsilon) \wedge g_2(\varepsilon) =$$

$$= (g_1\,g_2)(\varepsilon)$$

[1] The sets — written before and upper the matrices - are used to give the size and construction of the matrix. In this case the matrix $P(x)$ has $|S_1| + |S_2| + 1$ rows and columns and the construction is clear from its block form.

b) if $w = x \in X$ then

$$\chi_A(x) = \left[\mu_1 \,\vdots\, \mu_1\,\eta_1 \,\vdots\, 0\right] \begin{bmatrix} P_1(x) & P_1(x)\,\eta_1 & 0 \\ 0 & 0 & \mu_2 P_2(x) \\ 0 & 0 & P_2(x) \end{bmatrix} \begin{bmatrix} 0 \\ \mu_2\,\eta_2 \\ \eta_2 \end{bmatrix} =$$

$$= \left[\mu_1 P_1(x) \,\vdots\, \mu_1 P_1(x)\,\eta_1 \,\vdots\, \mu_1\,\eta_1\,\mu_2 P_2(x)\right] \begin{bmatrix} 0 \\ \mu_2\,\eta_2 \\ \eta_2 \end{bmatrix} =$$

$$= \mu_1\,\eta_1\,\mu_2 P_2(x)\,\eta_2 \vee \mu_1 P_1(x)\,\eta_1\,\mu_2\,\eta_2 = \big(g_1(\varepsilon)\wedge g_2(x)\big) \vee \big(g_1(x)\wedge g_2(\varepsilon)\big) =$$
$$= (g_1 g_2)(x)$$

c) if $w = x_1 x_2 \ldots . x_j \in X^*$, $j \geq 2$ then

$$\chi_A(w) = \left[\mu_1 \,\vdots\, \mu_1\,\eta_1 \,\vdots\, 0\right] \cdot$$

$$\cdot \begin{bmatrix} P_1(w) & P_1(w)\,\eta_1 & \bigvee_{l=1}^{j=1} P_1(x_1 \ldots x_l)\,\eta_1\,\mu_2 P_2(x_{l+1} \ldots x_j) \\ 0 & 0 & \mu_2 P(w) \\ 0 & 0 & P_2(w) \end{bmatrix} \begin{bmatrix} 0 \\ \mu_2\,\eta_2 \\ \eta_2 \end{bmatrix} =$$

$$= \left[\mu_1 P_1(w) \,\vdots\, \mu_1 P_1(w)\,\eta_1 \,\vdots\, \bigvee_{l=1}^{j-1} \mu_1 P_1(x_1 \ldots x_l)\,\eta_1\,\mu_2 P_2(x_{l+j} \ldots x_j) \vee \mu_1\,\eta_2 P_2(w)\right] \cdot$$

$$\cdot \begin{bmatrix} 0 \\ \mu_2\,\eta_2 \\ \eta_2 \end{bmatrix} = \bigvee_{uv=w} \big(\mu_1 P_1(u)\,\eta_1 \wedge \mu_2 P_2(v)\,\eta_2\big) = \bigvee_{uv=w} \big(g_1(u)\wedge g_2(v)\big) = g_1 g_2(w).$$

Hence $\chi_A = g_1 g_2$ that is $g_1 g_2$ is representable by A.

Lemma 5. $L(X)_{\mathrm{aut}}$ is closed under iteration.

PROOF. Let $g \in L(X)_{\mathrm{aut}}$. There is an L-fuzzy automaton $A = \langle S, X, P(x)_{x\in X}, \mu, \eta\rangle$ representing g. We construct the L-fuzzy automaton A' by

$$A' = \langle S \cup \{s\}, X, P'(x)_{x\in X}, \mu', \eta'\rangle \quad (s \notin S).$$

The block forms of $P'(x)$, μ', η' are:

$$P'(x) = \begin{array}{c} \\ s \\ S \end{array} \begin{array}{c} \begin{array}{cc} s & S \end{array} \\ \left[\begin{array}{c|c} \mu P(x)\eta & \mu P(x) \\ \hline P(x)\eta & P(x) \end{array}\right] \end{array}$$

$$\mu' = \overset{\quad s \quad S}{\left[\begin{array}{c|c} 1 & 0 \end{array}\right]} \text{ and } \eta'^T = \overset{\quad s \quad S}{\left[\begin{array}{c|c} 1 & 0 \end{array}\right]}.$$

Let us introduce the following notation for $w \in X^*$, $w \neq \varepsilon$:

$$R(w) = \sum_{\substack{u_1 \dots u_j = w \\ u_1, u_2, \dots, u_j \neq \varepsilon}} P(u_1)\,\eta\mu\, P(u_2)\,\eta\mu \dots \eta\mu\, P(u_j)$$

(We note that $R(x) = P(x)$).
It is clear that

$$R(wx) = \sum_{\substack{u_1 \dots u_j = w \\ u_1, u_2, \dots, u_j \neq \varepsilon}} P(u_1)\,\eta\mu\, P(u_2) \dots P(u_j x) +$$

$$+ \sum_{\substack{u_1 \dots u_j = w \\ u_1, u_2, \dots, u_j \neq \varepsilon}} P(u_1)\,\eta\mu\, P(u_2) \dots P(u_j)\,\eta\mu\, P(x) =$$

$$= R(w)\,P(x) + R(w)\,\eta\mu\, P(x)$$

(We used the equation $P(u_j x) = P(u_j)\,P(x)$ and the distributive property.) Using this recurrent formula we prove the following statement by induction on the length of w.

Statement 1. For $w \in X^*$, $w \neq \varepsilon$ holds

$$P'(w) = \left[\begin{array}{c|c} \mu R(w)\eta & \mu R(w) \\ \hline R(w)\eta & R(w) \end{array}\right].$$

PROOF.

For $|w| = 1$ it is true by definition. Let $w' = wx$, then

$$P'(w') = P'(w)P'(x) = \left[\begin{array}{c|c} \mu R(w) & \mu R(w) \\ \hline R(w)\eta & R(w) \end{array}\right] \left[\begin{array}{c|c} \mu P(x) & \mu P(x) \\ \hline P(x)\eta & P(x) \end{array}\right] =$$

$$= \left[\begin{array}{c|c} \mu(R(w)\,\eta\mu\, P(x) + R(w)\,P(x))\,\eta & \mu(R(w)\,\eta\mu\, P(x) + \mu\, R(w)\,P(x)) \\ \hline (R(w)\,\eta\mu\, P(x) + R(w)\,P(x))\,\eta & R(w)\,\eta\mu\, P(x) + R(w)\,P(x) \end{array}\right] =$$

$$= \left[\begin{array}{c|c} \mu R(w')\eta & \mu R(w') \\ \hline R(w')\eta & R(w') \end{array}\right],$$ so Statement 1 is proved.

We continue the proof of Lemma 5.
Let us calculate $\chi_{A'}(w)$ $w \in X^*$.
For $w = \varepsilon$ $\chi_{A'}(\varepsilon) = \mu' \eta' = 1 = g^*(\varepsilon)$.
For $w \neq \varepsilon$

$$\chi_{A'}(w) = \mu' P'(w) \eta' = \left[1 \,\vdots\, 0 \right] \left[\begin{array}{c|c} \mu R(w) \eta & \mu R(w) \\ \hline R(w) \eta & R(w) \end{array} \right] \left[\begin{array}{c} 1 \\ \hline 0 \end{array} \right] = \mu R(w) \eta.$$

Using the definition of $R(w)$ we get

$$\chi_{A'}(w) = \bigvee_{\substack{u_1 \dots u_j = w \\ u_1, \dots, u_j \in X^+}} g(u_1) \wedge g(u_2) \dots g(u_j) = g^*(w).$$

It means, that $\chi_{A'} = g^*$, therefore g^* is representable by A'.

4. Analysis of L-fuzzy automata

Our method is a generalization of Kleene's classical analysis method.

Definition 7. Let an L-fuzzy automaton

$$A = \langle S, X, P(x), \mu, \eta \rangle \; (S = \{s^1, \dots, s^n\} \text{ and } X = \{x^1, \dots, x^k\})$$

be given.
The oriented graph G_A of A is defined by the following. The points of G_A are the states of A. For $s, s' \in S$ and $x \in X$ there is an arrow from s to s' with the label $(x, P(x)(s, s'))$, representing one of the possible state-transitions on the effect of input x. $P(x)(s, s')$ is the fuzzy-measure of this transition.

The state transition s, s' on the effect of input word $w \in X^+$ can be characterized by the paths α, whose startpoint is s, endpoint is s' and going from s to s' along α we can read the word w. Our goal is to define an L-fuzzy event f_α for each transition path α. Let us introduce the following notations

a) $st(\alpha) = s$, $end(\alpha) = s'$
b) $sr(\alpha) = s_0, \dots, s_m$ is the series of points in α.

It is clear that $s_0 = s$ and $s_m = s'$.

c) $wd(\alpha) = x_1 \dots x_m$ is the word readable along α.

Definition 8. We say, that α is an empty path if $sr(\alpha)$ consists of a single element.

For an empty path α we have $wd(\alpha) = \varepsilon$.

Using this notations we can define f_α, the L-fuzzy event, represented by the state transition α.

Definiton 9.
If α is empty, then $f_\alpha = \varepsilon$ else

$$f_\alpha(w) = \begin{cases} \bigwedge_{i=1}^{m} P(x_i)(s_{i-1}, s_i) & \text{if } w = wd(\alpha) \\ 0 & \text{else.} \end{cases}$$

(It is clear that if α is the composition of paths

$$\alpha_1, \alpha_2, \ldots, \alpha_j \text{ then } wd(\alpha) = wd(\alpha_1)\, wd(\alpha_2) \ldots wd(\alpha_j)$$

and

$$f_\alpha = f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_j}.)$$

Definiton 10. The fuzzy event represented by A with the initial state s and final state s' is

$$A_{s,s'} = \bigcup_{\substack{st(\alpha)=s \\ \mathrm{end}(\alpha)=s'}} f_\alpha .$$

It is clear from the definiton f_α that

$$A_{s,s'}(w) = \bigvee_{\substack{wd(\alpha)=w \\ st(\alpha)=s,\ \mathrm{end}(\alpha)=s,}} f_\alpha(w).$$

From the definiton of matrix multiplication it follows, that

$$A_{s,s'}(w) = P(w)(s, s').$$

If we can prove the L-regularity of $A_{s,s'}$, then by using the closure properties of $L(X)_{\mathrm{reg}'}$, from

$$\chi_A = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \mu_i A_{s^i, s^j} \eta_j$$

we get the L-regularity of χ_A.

Lemma 6. A_{s^i, s^j} is L-regular for all $s^i, s^j \in S$.

PROOF. Let us introduce the following notations ($l \geq 0$):

$$S^l = \{s^r;\ 1 \leq r \leq \min(n, l)\},\ (S^0 = \emptyset)$$

$$\Gamma^l_{i,j} = \left\{ \alpha, \begin{array}{l} \mathrm{st}(\alpha) = s^i \ \mathrm{end}(\alpha) = s^j \text{ and the inner letters} \\ \text{of } sr(\alpha) \text{ are elements of } S^l \end{array} \right\},$$

$$A^l_{i,j} = \bigcup_{\alpha \in \Gamma_{i,j}} f_\alpha .$$

It is sufficient to prove the L-regularity of A^l_{ij} for because $A_{s^i,s^j} = A^n_{i,j} = A^{n+1}_{i,j} = \ldots$. We do this by induction on l.

a) For $l = 0$

$$A^0_{i,j} = \bigcup_{\substack{\mathrm{st}(\alpha)=s^i,\ \mathrm{end}(\alpha)=s^j, \\ wd(\alpha) \in X \cup \{\varepsilon\}}} f_\alpha .$$

It is clear from definition of G_A, $P(\varepsilon)$ and f_α that

$$f_\alpha = P(wd(\alpha))\,(s^i, s^j)\, wd(\alpha).$$

Hence $wd(\alpha)\in X\cup\{\varepsilon\}$ is an elementary event, therefore its product with the scalar $P(wd(\alpha))\,(s^i, s^j) - f_\alpha$ — is L-regular. But $L(X)_{reg}$ closed under finite union, thus $A^0_{i,j}\in L(X)_{reg}$ too.

b) Let us suppose, that $A^l_{i,j}\in L(X)_{reg}$ for all $1\le i, j\le n$. We must prove the L-regularity of $A^{l+1}_{i,j}$.

A path $\alpha\in\Gamma^{l+1}_{i,j}$ can be decomposed into three, possibly empty, subpaths $\alpha', \alpha'', \alpha'''$ where $\alpha'\in\Gamma^l_{i,l+1}$, $\alpha''\in\Gamma^{l+1}_{l+1,l+1}$, $\alpha'''\in\Gamma^l_{l+1,j}$. (We get α in such a way that we go along the path α up to the first occurence of the state s^{l+1}. Going from the first occurence of s^{l+1} up to the last occurence of s^{l+1} we get α'' and if we go from the last occurence of s^{l+1} up to the end of α we get α'''.) If $\alpha'\in\Gamma^l_{i,l+1}$, $\alpha''\in\Gamma^{l+1}_{l+1,l+1}$, $\alpha'''\in\Gamma^l_{l+1,j}$ are arbitrary paths, then its composition is an element of $\Gamma^{l+1}_{i,j}$. There fore we can write:

$$A^{l+1}_{i,j} = \bigcup_{\alpha\in\Gamma^{l+1}_{i,j}} f_\alpha = \bigcup_{\substack{\alpha'\in\Gamma^l_{i,l+1}\\ \alpha''\in\Gamma^{l+1}_{l+1,l+1}\\ \alpha'''\in\Gamma^l_{l+1,j}}} f_{\alpha'} f_{\alpha''} f_{\alpha'''} =$$

$$= \Big(\bigcup_{\alpha'\in\Gamma^l_{i,l+1}} f_{\alpha'}\Big)\Big(\bigcup_{\alpha''\in\Gamma^{l+1}_{l+1,l+1}} f_{\alpha''}\Big)\Big(\bigcup_{\alpha'''\in\Gamma^l_{l+1,j}} f_{\alpha'''}\Big) = A^l_{i,l+1}\, A^{l+1}_{l+1,l+1}\, A^l_{l+1,j}\,.$$

A nonempty path $\alpha''\in\Gamma^{l+1}_{l+1,l+1}$ can be decomposed on the natural way into a finite number of paths $\alpha_1, \alpha_2, \ldots, \alpha_q\in\Gamma^l_{l+1,l+1}$ and for arbitrary $q>0$ and for arbitrary nonempty paths $\alpha_1, \ldots, \alpha_q\in\Gamma^l_{l+1,l+1}$ their composition is a nonempty element of $\Gamma^{l+1}_{l+1,l+1}$.

Thus

$$A^{l+1}_{l+1,l+1} = \varepsilon \bigcup_{\substack{\alpha\in\Gamma^{l+1}_{l+1,l+1}\\ \alpha \text{ nonempty}}} f_\alpha = \varepsilon \bigcup_{\substack{\alpha=\alpha_1\ldots\alpha_q\\ \alpha_1,\ldots,\alpha_q\in\Gamma^l_{l+1,l+1} \text{ nonempty}}} f_{\alpha_1}\cdots f_{\alpha_q} =$$

$$= \varepsilon \cup \Bigg(\bigcup_{q=1}^{\infty} \prod_{\substack{s=1\\ \alpha_s\in\Gamma^l_{l+1,l+1} \text{ nonempty}}}^{q} f_{\alpha_s}\Bigg) = \varepsilon \cup \Bigg(\bigcup_{q=1}^{\infty} (A^l_{l+1,l+1})^q\Bigg) = (A^l_{l+1,l+1})^*\,.$$

Hence

$$A^{l+1}_{i,j} = A^l_{i,l+1}(A^l_{l+1,l+1})^*\, A^l_{l+1,j}\in L(X)_{reg}\,.$$

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