COMPUTATION OF THE DETERMINANT OF FIVE DIAGONAL SYMMETRIC TOEPLITZ MATRICES

By

I. KÁTAI and E. RAHMY

(Received September 29, 1978)

1. We shall consider the problem of finding the determinant of symmetric band Toeplitz matrices with five non-zero diagonals. Matrices of this structure occur in many fields of numerical analysis. Our method is simple quite enough and has the advantage that the time of computation is short, especially for matrices having large order.

2. Computation the determinant of general five diagonal band matrices

Let A_n denote the following square matrix:

$$(2.1)\qquad A_n = \begin{bmatrix} a_1 & b_1 & c_1 & & & \\ z_1 & a_2 & b_2 & c_2 & 0 & \\ y_1 & z_2 & a_3 & b_3 & c_3 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & \cdot \\ & & & & & c_{n-2} \\ & & & & & b_{n-1} \\ & & \cdot & \cdot & & \\ 0 & & y_{n-2} & z_{n-1} & & a_n \end{bmatrix}.$$

Allgover [1] derived the following recursion formula for the computation of the determinant of A_n.

Let $D_n = det\ A_n$, B_{n-1} and C_{n-1} denote the minors of b_{n-1} and c_{n-1} in A_n, respectively. Let Y_{n-2} be the minor of y_{n-1} in B_{n-1}. Similarly, let X_{n-1}

be the minor of y_{n-2} in C_{n-1}, E_{n-2} be the minor of c_{n-3} in X_{n-2}. We get the following relations:

$$\begin{aligned}
D_n &= a_n D_{n-1} - b_{n-1} B_{n-2} + c_{n-2} C_{n-1} \\
B_{n-1} &= z_{n-1} D_{n-2} - y_{n-2} Y_{n-2} \\
Y_{n-2} &= b_{n-2} D_{n-3} - c_{n-3} B_{n-3} \\
C_{n-1} &= z_{n-1} B_{n-2} - y_{n-2} X_{n-2} \\
X_{n-2} &= a_{n-1} D_{n-3} - c_{n-3} E_{n-3} \\
E_{n-3} &= y_{n-3} D_{n-4}.
\end{aligned}$$

These relations hold for $n \geq 1$, if we define $b_\nu, c_\nu, z_\nu, a_\nu, y_\nu$ to be zero for non-positive ν. Let V_n denote the vectorial

(2.2) $$\mathbf{V}_n = [D_n, B_n, C_n, Y_n, X_n, E_n]^T,$$

and $M(n)$ be the matrix

(2.3) $$M(n) = \begin{bmatrix} a_n & -b_{n-1} & c_{n-2} & 0 & 0 & 0 \\ z_n & 0 & 0 & -y_{n-1} & 0 & 0 \\ 0 & z_n & 0 & 0 & -y_{n-1} & 0 \\ b_n & -c_{n-1} & 0 & 0 & 0 & 0 \\ a_{n+1} & 0 & 0 & 0 & 0 & -c_{n-1} \\ y_n & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, by taking $\mathbf{V}_0 = [1, 0, 0, 0, 0, 0]^T$ and all the undefined coefficients to be equal zero, we get

(2.4) $$\mathbf{V}_n = M(n)\,\mathbf{V}_{n-1}$$

for $n \geq 1$.

We remark that the relation (2.4) gives a possibility to compute D_n with $O(\log n)$ operations in a computer having as many as need parallel processors. Without parallel computation we can determine D_n with $O(n)$ operations.

3. Simple recursion for the determinant of symmetric Toeplitz matrices

Let

$$R_n = \begin{bmatrix} a_0 & a_1 & a_2 & & & \\ a_1 & a_0 & a_1 & a_2 & & \\ a_2 & a_1 & a_0 & a_1 & a_2 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & \cdot \\ & & & & & a_2 \\ & & & & & a_1 \\ & & & a_2 & a_1 & a_0 \end{bmatrix}$$

be our matrix of order n. Observing that $B_{n-1} = Y_{n-1}$, from the relations (2.3), (2.4) we get the following recursion formula. Let

$$N = \begin{bmatrix} a_0 & -a_1 & a_2 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & -a_2 & 0 \\ a_0 & 0 & 0 & 0 & -a_2 \\ a_2 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2}$$

and

$$U_n = [D_n, B_n, C_n, X_n, E_n]^T. \tag{3.3}$$

Then

$$U_n = NU_{n-1} \tag{3.4}$$

for $n \geq 1$. We set $U_n = [1, 0, 0, 0, 0]^T$, and get

$$U_n = N^n\, U_0. \tag{3.5}$$

In what follows we shall assume that $a_2 = 1$.

4. Substitution of N^n by another polynomial of N

Let

$$S(\lambda) = \det\ (N - \lambda I) \tag{4.1}$$

be the characteristic polynomial of N. By an easy computation we get

$$S(\lambda) = (1-\lambda)\, Q(\lambda), \tag{4.2}$$

where

$$Q(\lambda) = 1 + (2-a_0)\,\lambda + (2 + a_1^2 - 2a_0)\,\lambda^2 + (2-a_0)\,\lambda^3 + \lambda^4. \tag{4.3}$$

Observing the symmetry of the coefficients of $Q(\lambda)$, we can determine its roots easily.
Putting

$$w = \lambda + \lambda^{-1},$$

we get

$$\varphi(w) = \frac{Q(\lambda)}{\lambda^2} = w^2 + (2-a_0)\,w + (a_1^2 - 2a_0).$$

Let w_1, w_2 be the roots of $\varphi(w) = 0$:

$$w_j = \frac{-(2-a_0) \pm \sqrt{(2-a_0)^2 - 4(a_1^2 - 2a_0)}}{2} \quad (j = 1,2). \tag{4.4}$$

Then the roots of $Q(\lambda)$ are Θ_j, Θ_j^{-1} $(j = 1, 2)$, where

$$(4.5) \qquad \begin{cases} \Theta_j = \dfrac{w_j + \sqrt{w_j^2 - 4}}{2}, \\ \Theta_j^{-1} = \dfrac{w_j^2 - \sqrt{w_j^2 - 4}}{2}. \end{cases}$$

Divide z^n by $S(z)$. We get

$$(4.6) \qquad z^n = k(z)\ S(z) + r(z),$$

where $r(z)$ is a polynomial of degree at most 4.
From the Cayley-Hamilton theorem we get that $S(N) = O$, whence $N^n = r(N)$, and so

$$(4.7) \qquad U_n = r(N)\ U_0.$$

After we have found the coefficients of $r(z)$,

$$(4.8) \qquad r(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4,$$

we get

$$(4.9) \qquad D_n = A_0 + A_1 D_1 + A_2 D_2 + A_3 D_3 + A_4 D_4.$$

We can compute that

$$(4.1) \qquad \begin{aligned} D_1 &= a_0 \\ D_2 &= a_0^2 - a_1^2 \\ D_3 &= a_0^3 - a_0 + 2(1 - a_0) a_1^2 \\ D_4 &= a_0 D_3 - a_1^2 [a_0^2 - a_1^2 + 2(1 - a_0)] + (1 - a_0^2). \end{aligned}$$

First we decide the cases in which $S(z)$ has multiple roots.

A) $w_1 = w_2 = 2$. Then $a_0 = 6$, $a_1 = \pm 4$, $S(\lambda) = (1-\lambda)^5$.

B) $w_1 = 2, w_2 = -2$. Then $a_0 = 2$, $a_1 = 0$, $S(\lambda) = (1-\lambda)^3 \cdot (1+\lambda)^2$.

C) $w_1 = 2$, $w_2 \neq \pm 2$, Then $a_0 = \dfrac{a_1^2}{4} + 2$, $a_0 \neq 6{,}2$,

$$S(\lambda) = (1-\lambda)^3 (\lambda - \Theta_2)(\lambda - \Theta_2^{-1}).$$

D) $w_1 = -2, w_2 \neq \pm 2$. Then $a_1 = 0$, $a_0 \neq \pm 2$,

$$S(\lambda) = (1-\lambda)(\lambda+1)^2(\lambda - \Theta_2)(\lambda - \Theta_2^{-1}).$$

E) $w_1 = w_2 \neq \pm 2$. Then $a_1 = \pm\frac{a_0+2}{2}$, $a_0 \neq 6$, $a_0 \neq -2$,

$$S(\lambda) = (1-\lambda)(\lambda-\Theta_1)^2(\lambda-\Theta_1^{-1})^2.$$

F) $w_1 = w_2 = -2$. Then $a_0 = -2$, $a_1 = 0$,

$$S(\lambda) = (1-\lambda)\cdot(\lambda+1)^4,$$

In another choosing of a_0, a_1 all of the roots of $S(z)$ are distinct. We compute $r(z)$ as an interpolation polynomial. For a root Θ of $S(z)$ we get: $\Theta^n = r(\Theta)$, and a similar formula holds for the derivative, if Θ is a multiple root of $S(z)$.

5. The case of simple roots

Suppose that all roots of $S(z)$ are simple. From (4.6) we have

$$\begin{cases} 1 = r(1), \\ \Theta_j^{\pm n} = r(\Theta_j^{\pm 1}) \quad (j = 1,2). \end{cases} \tag{5.1}$$

These relations determine the coefficients of $r(z)$. We put it in the form

$$\begin{cases} r(z) = (z-1)\,H(z) + AQ(z), \\ \qquad A = 1/Q(1). \end{cases} \tag{5.2}$$

Let

$$H(z) = \sum_{i=0}^{3} y_i \cdot z^i. \tag{5.3}$$

From (5.1) it follows that

$$\frac{\Theta_j^{\pm n}}{\Theta_j^{\pm 1}-1} = H(\Theta_j^{\pm 1}) \quad (j = 1,2),$$

whence the relations

$$\begin{bmatrix} 1 & \Theta_1 & \Theta_1^2 & \Theta_1^3 \\ 1 & \Theta_1^{-1} & \Theta_1^{-2} & \Theta_1^{-3} \\ 1 & \Theta_2 & \Theta_2^2 & \Theta_2^3 \\ 1 & \Theta_2^{-1} & \Theta_2^{-2} & \Theta_2^{-3} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f(\Theta_1) \\ f(\Theta_1^{-1}) \\ f(\Theta_2) \\ f(\Theta_2^{-1}) \end{bmatrix} \tag{5.4}$$

and

$$f(\tau) = \frac{\tau^n}{\tau-1} \tag{5.5}$$

must hold.

Let K denote the matrix at the left hand side of (5.4). Let furthermore σ_h denote the h'-th powersum of the roots of $Q(z)$:

$$\sigma_h = \Theta_1^h + \Theta_1^{-h} + \Theta_2^h + \Theta_2^{-h}.$$

We multiply the relation (5.4) by the transpose of K, we get

$$K^T K \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = K^T \begin{bmatrix} f(\Theta_1) \\ f(\Theta_1^{-1}) \\ f(\Theta_2) \\ f(\Theta_2^{-1}) \end{bmatrix}.$$

We can see easily that

$$K^T K = \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \\ \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \end{bmatrix} \left(\stackrel{\text{def}}{=} T\right),$$

and so that

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = T^{-1} K^T \begin{bmatrix} f(\Theta_1) \\ f(\Theta_1^{-1}) \\ f(\Theta_2) \\ f(\Theta_2^{-1}) \end{bmatrix}. \tag{5.6}$$

We can compute T^{-1} as a rational function of a_0, a_1, by using the Newton-Girard formulas for σ_h. The computation of y_j by (5.6) is more stable than by (5.4).

By the relations (5.2), (4.8), (4.9), (4.10) we can compute D_n immediately.

6. The case of multiple roots

A) $a_0 = 6$, $a_1 = \pm 4$. Then $S(z) = (1-z)^5$, and from (4.6) we get

$$\frac{r^k(1)}{k!} = \binom{n}{k} \quad (k = 0, 1, 2, 3, 4).$$

By using the representation

$$r(z) = \sum_{k=0}^{4} \frac{r^{(k)}(1)}{k!} (z-1)^k,$$

and that (see (4.10): $D_1 = 6$, $D_2 = 20$, $D_3 = 50$, $D_4 = 105$,

$$D_n = A_0 + 6\,A_1 + 20\,A_2 + 50\,A_3 + 105\,A_4,$$

where

$$A_0 = 1-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4},$$

$$A_1 = \binom{n}{1}-2\binom{n}{2}+3\binom{n}{3}-4\binom{n}{4},$$

$$A_2 = \binom{n}{2}-3\binom{n}{3}+6\binom{n}{4}.$$

$$A_3 = \binom{n}{3}-4\binom{n}{4},$$

$$A_4 = \binom{n}{4}.$$

B) $a_0 = 2$, $a_1 = 0$. Then $S(z) = (1-z)^3(1+z)^2$,

(6.1)
$$\begin{cases} \dfrac{r^{(k)}(1)}{k!} = \dbinom{n}{k} \ (k = 0, 1, 2) \\ r(-1) = (-1)^n, r'(-1) = -n(-1)^n. \end{cases}$$

Now we have

$$D_1 = 2, D_2 = 4, D_3 = 6, D_4 = 9.$$

Putting

$$r(z) = A+B(z-1)+C(z-1)^2+(z-1)^3[\alpha(z+1)+\beta],$$

we get

$$D_n = A+B+C+4\alpha-\beta.$$

Furthermore, from (6.1) we get

$$A = 1, B = n, C = \binom{n}{2}, (-1)^n = A-2B+4C-8\beta,$$

$$n(-1)^{n-1} = B-4C+12\beta-8\alpha.$$

Hence

(6.2)
$$\begin{cases} \beta = \dfrac{2n^2+[1-(-1)^n]}{8}, \\ \alpha = \dfrac{12\beta-2n^2-n[1-(-1)^n]}{8}. \end{cases}$$

So we get

$$D_n = 1+n+\binom{n}{2}+\frac{n^2}{4}+\left(\frac{5}{4}-n\right)\frac{1-(-1)^n}{2}.$$

C) $a_0 = \dfrac{a_1^2}{4}+2$, $a_1 \neq \pm 4{,}0$. Then $S(z) = (1-z)^3(z-\Theta)(z-\Theta^{-1})$, $\Theta \neq \pm 1$.
Now we take $r(z)$ in the form

$$r(z) = \sum_{k=0}^{2} \frac{r^{(k)}(1)}{k!}(z-1)^k+(z-1)^3(\alpha z+\beta).$$

We have

$$r(1) = 1,\quad r'(1) = n,\quad \frac{r''(1)}{2!} = \binom{n}{2},$$

$$\Theta^n = r(\Theta),\ \Theta^{-n} = r(\Theta^{-1}).$$

From the last two relations we get

$$\alpha = \frac{t(\Theta)-t(\Theta^{-1})}{\Theta-\Theta^{-1}},\quad \beta = \frac{\Theta\cdot t(\Theta^{-1})-\Theta^{-1}\cdot t(\Theta)}{\Theta-\Theta^{-1}},$$

where

$$t(\Theta) = \frac{\Theta^n-1-n(\Theta-1)-\binom{n}{2}(\Theta-1)^2}{(\Theta-1)^3}.$$

After we have computed Θ, $t(\Theta)$, $t(\Theta^{-1})$, α, β, by (4.10), (4.9) we obtain D_n.
D) $a_1 = 0$, $a_0 \neq \pm 2$. Then $S(z) = (1-z)(z+1)^2(z-\Theta)(z-\Theta^{-1})$.
Now we take

$$r(z) = r(-1)+r'(-1)(z+1)+(z+1)^2[\alpha+\beta z+\gamma z^2].$$

Observing that

$$r(-1) = (-1)^n,\ r'(-1) = (-1)^{n-1}n,\ r(1) = 1$$

$$r(\Theta) = \Theta^n,\ r(\Theta^{-1}) = \Theta^{-n} \qquad (\Theta \neq \pm 1),$$

the coefficients α, β, γ are easily computable.
E) $a_1 = \pm\dfrac{a_0+2}{2}$, $a_0 \neq 6, -2$. Then $S(z) = (1-z)(z-\Theta)^2(z-\Theta^{-1})^2$, $\Theta \neq \pm 1$.
In this case

$$r(1) = 1,\ r(\Theta) = \Theta^n,\ r(\Theta^{-1}) = \Theta^{-n},$$

$$r'(\Theta) = n\Theta^{n-1},\ r'(\Theta^{-1}) = n\cdot\Theta^{-(n-1)}.$$

The coefficients A_0, A_1, A_2, A_3, A_4 of $r(z)$ in (4.8) can be computed from the linear equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \Theta & \Theta^2 & \Theta^3 & \Theta^4 \\ 1 & \Theta^{-1} & \Theta^{-2} & \Theta^{-3} & \Theta^{-4} \\ 0 & 1 & 2\Theta & 3\Theta^2 & 4\Theta^3 \\ 0 & 1 & 2\Theta^{-1} & 3\Theta^{-2} & 4\Theta^{-3} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \Theta^n \\ \Theta^{-n} \\ n \cdot \Theta^{n-1} \\ n \cdot \Theta^{-(n-1)} \end{bmatrix}.$$

F) $a_0 = -2$, $a_1 = 0$. Then $S(z) = (1-z)(z+1)^4$.
We take

$$r(z) = r(-1) + r'(-1)(z+1) + \frac{r''(-1)}{2!}(z+1)^2 + \frac{r'''(-1)}{3!}(z+1)^3 + \\ + \varkappa(z+1)^4.$$

Now we have

$$\frac{r^{(k)}(-1)}{k!} = (-1)^{n-k}\binom{n}{k} \quad (k = 0, 1, 2, 3),$$

$$r(1) = 1.$$

Hence we get

$$\varkappa = \frac{1}{16}\left\{1 - (-1)^n\left[1 - 2\cdot\binom{n}{1} + 4\cdot\binom{n}{2} - 8\cdot\binom{n}{3}\right]\right\}.$$

Furthermore $D_1 = -2$, $D_2 = 4$, $D_3 = -6$, $D_4 = 9$, and so

$$D_n = (-1)^n\left[1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3}\right] + 2\varkappa.$$

REFERENCE

[1] *Allgover E. L.*, Criteria for positive definiteness of some-band matrices, Numer. Math., *16*, 157–162, 1970

Eötvös Loránd's University
Department of Numerical Analysis and Computer Science
1088 Budapest, Múzeum krt. 6–8.