

SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS, V.

APPLICATION WITH PROGRAMS TO THE METHOD

By

THARWAT FAWZY, JÁNOS KŐHEGYI and ISTVÁN FEKETE

(Received November 26, 1976)

We have written a program for the ICL 1900 (ODRA 1304) Computer in FORTRAN IV for the method described in [2], [3], [4] and [5] to solve the following examples:

Example 1.

As an application to the approximate solution of the first order differential equation $y' = f(x, y)$, we choose the case when $f \in C^0$ and for this purpose we consider the differential equation

$$y' = 1.5 x^{1/2} y \quad \text{with} \quad y(0) = 1.$$

The exact solution is obviously

$$y(x) = e^{x^{1.5}}.$$

Equation (2.1.5), presented in [4], gives the approximate value of $y(x)$ at x_{k+1} and it is as follows:

$$\bar{y}_{k+1} = \bar{y}_k + (\bar{y}_k - x_k \bar{y}'_k) (x_{k+1}^{1.5} - x_k^{1.5}) + 0.6 \bar{y}'_k (x_{k+1}^{2.5} - x_k^{2.5})$$

and the formula (2.2.9), presented in [4], gives

$$\bar{y}'_{k+1} = f(x_{k+1}, \bar{y}_{k+1}) = 1.5 x_{k+1} \bar{y}_{k+1}$$

The chosen interval is $0 \leq x \leq 1$ and the step h has been chosen to be 0.001.

The flow chart diagram and the program giving the numerical and exact results are presented in flowchart 1, program 1 and results 1.

Example 2.

As an application to the approximate solution of the second order differential equation $y'' = f(x, y, y')$ we consider in this example the case when $f \in C^0$ and so we consider the differential equation

$$y'' = \left(\frac{25}{4} x^3 + \frac{15}{4} x^{0.5} \right) y \quad \text{with } y(0) = 1 \text{ and } y'(0) = 0$$

and in this case the choosen interval is $0 \leq x \leq 1$.
Obviously the exact solution is

$$y = e^{x^{2.5}}.$$

Equations (2.1.11), (2.1.12) and (2.1.13), presented in [3], give the approximate values of $y(x)$, $y'(x)$ and $y''(x)$ at $x = x_{k+1}$ and they are as follow:

$$\bar{y}_{k+1} = \bar{y}_k + \bar{y}'_k h + A_k(G_{k+1} - G_k) + B_k(Q_{k+1} - Q_k) + C_k(T_{k+1} - T_k) - h(A_k D_k + B_k E_k + C_k F_k),$$

$$\bar{y}'_{k+1} = \bar{y}'_k + A_k(D_{k+1} - D_k) + B_k(E_{k+1} - E_k) + C_k(F_{k+1} - F_k)$$

and

$$\bar{y}''_{k+1} = \left(\frac{25}{4} x_{k+1}^3 + \frac{15}{4} x_{k+1}^{0.5} \right) \bar{y}_{k+1}$$

where,

$$\begin{aligned} A_k &= \bar{y}_k - x_k \bar{y}'_k + 0.5 x_k^2 \bar{y}''_k && \text{with } A_0 = 1, \\ B_k &= \bar{y}'_k - x_k \bar{y}''_k && \text{with } B_0 = 0, \\ C_k &= 0.5 \bar{y}''_k && \text{with } C_0 = 0, \\ D_k &= \frac{25}{16} x_k^4 + \frac{5}{2} x_k^{1.5} && \text{with } D_0 = 0, \\ E_k &= \frac{5}{4} x_k^5 + 1.5 x_k^{2.5} && \text{with } E_0 = 0, \\ F_k &= \frac{25}{24} x_k^6 + \frac{15}{14} x_k^{3.5} && \text{with } F_0 = 0, \\ G_k &= \frac{5}{16} x_k^5 + x_k^{2.5} && \text{with } G_0 = 0, \\ Q_k &= \frac{5}{24} x_k^6 + \frac{3}{7} x_k^{3.5} && \text{with } Q_0 = 0 \text{ and} \\ T_k &= \frac{25}{128} x_k^7 + \frac{15}{63} x_k^{4.5} && \text{with } T_0 = 0. \end{aligned}$$

The flow chart diagram, the program and the results are given in flow-chart 2, program 2 and results 2. In this example two numerical results are given in order to show the influence of h on the results.

Example 3.

As an application to the case $f \in C^\infty$, we solve the differential equation

$$y'' = y \quad \text{with } y(0) = 2 \quad \text{and } y'(0) = 0.$$

Obviously $f \in C^\infty$ implies $f \in C^r$ where $r \in I^+$.

Our interval will be $0 \leq x \leq 1$ and the method is applied to give the approximate values as follow:

$$\bar{y}_{k+1} = \bar{y}_k + \bar{y}'_k h + \sum_{j=0}^{r+2} \frac{\bar{y}''_k}{(j+2)!} h^{j+2},$$

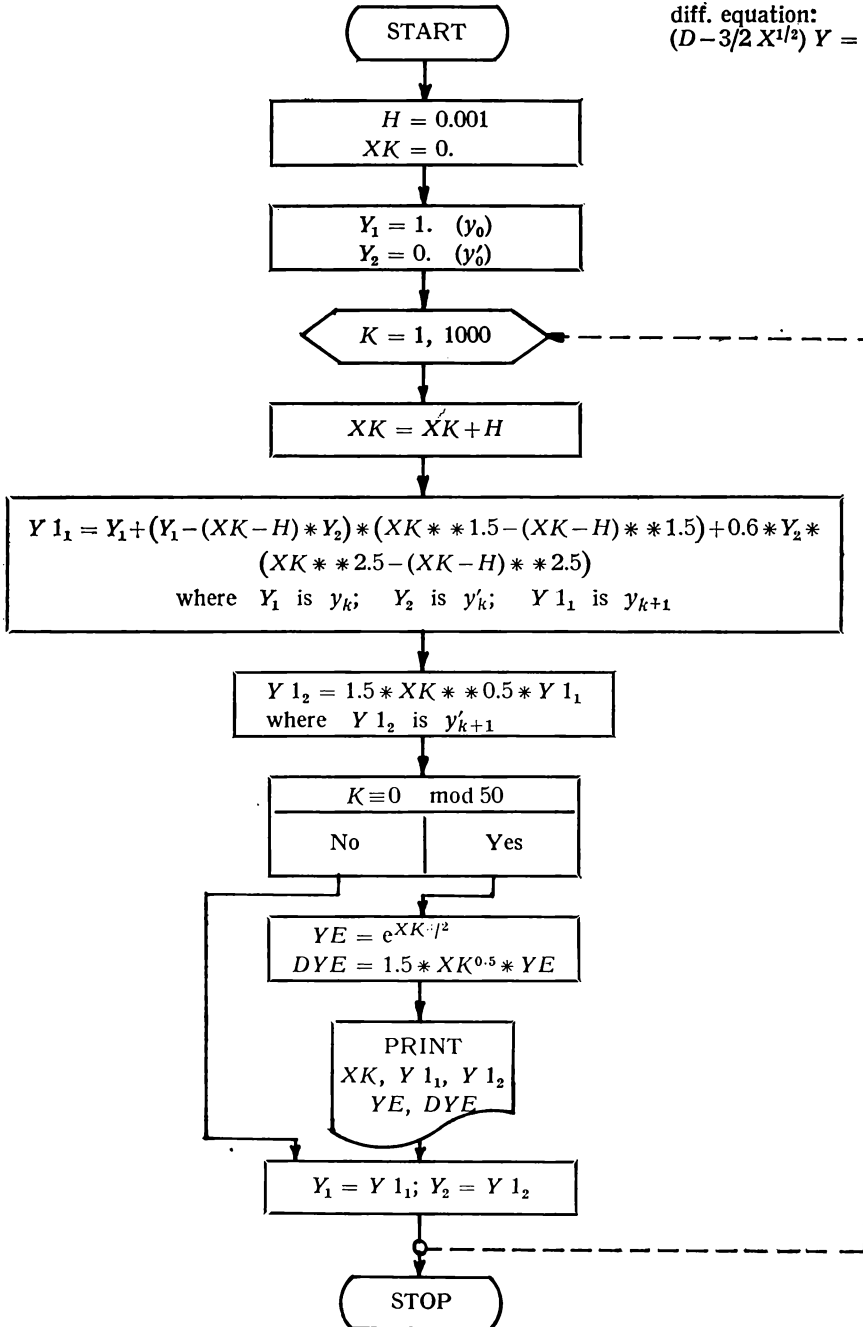
$$\bar{y}'_{k+1} = \bar{y}'_k + \sum_{j=0}^{r+2} \frac{\bar{y}''_k}{(j+1)!} h^{j+1} \quad \text{and}$$

$$\bar{y}''_{k+1} = \bar{y}''_k.$$

The flow chart diagram, program and results are presented in flowchart 3, program 3./a, program 3./b, results 3./a and results 3./b. In the first case 3./a we have choosen r to be $r = 2$. In the second case 3./b r is equal to zero. If r is larger than 2, the results will be more accurate as it was shown in the convergence theorem, (see reference [3]), the error is $O(h^{r+2})$.

Flowchart 1.

diff. equation:
 $(D - 3/2 X^{1/2}) Y = 0$



```

MASTER SPLINE
C**** APPROXIMATE SOLUTION OF THE DIFF. EQUATION
C**** (D - 1.5*X**0.5)* Y = 0
C**** BY SPLINE FUNCTIONS TECHHIQUE
DIMENSION Y(2),Y1(2)
WRITE (2,101)
101 FORMAT(1H1)
H=0.001
XK=0.
Y(1)=1.
Y(2)=0.
DO 1 K=1,1000
XK=XK+H
Y1(1)=Y(1)+(Y(1)-(XK-H)*Y(2))*(XK**1.5-(XK-H)**1.5)+0.6*Y(2)*
1(XK**2.5-(XK-H)**2.5)
Y1(2)=1.5*XK**0.5*Y1(1)
IF (K-K/50*50) 2,0,2
YE=EXP(XK**1.5)
DYE=1.5*XK**0.5*YE
WRITE (2,100) XK,Y1(1),Y1(2),YE,DYE
100 FORMAT(1H0,12HNUM. RESULT,2X,2HX=,F4.2,2X,2HY=,E18.10,2X,
13HDY=,E18.10/13H EXACT RESULT,12X,E18.10,5X,E18.10)
2 Y(1)=Y1(1)
1 Y(2)=Y1(2)
STOP
END

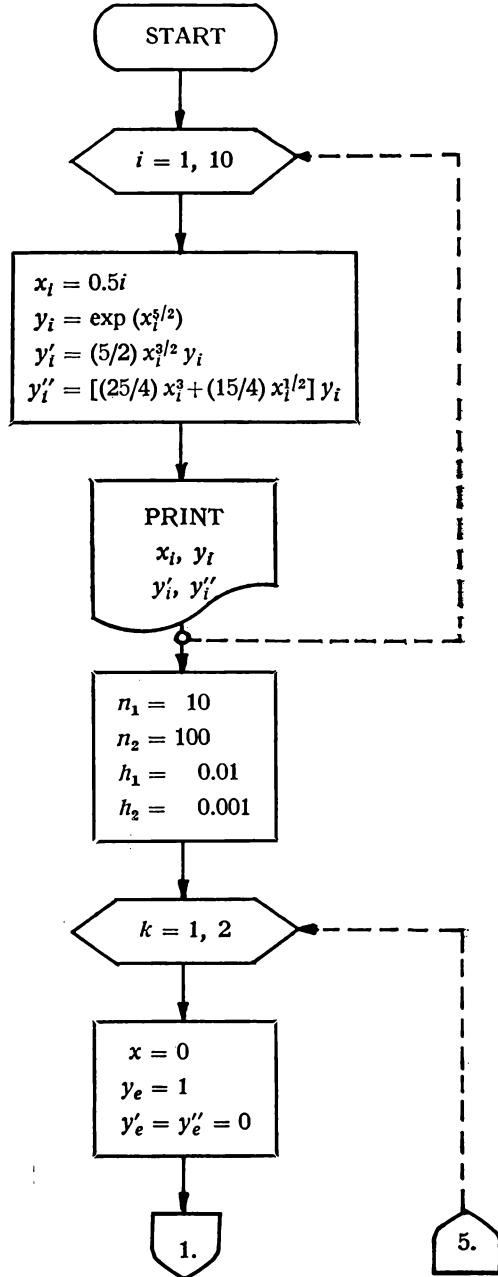
```

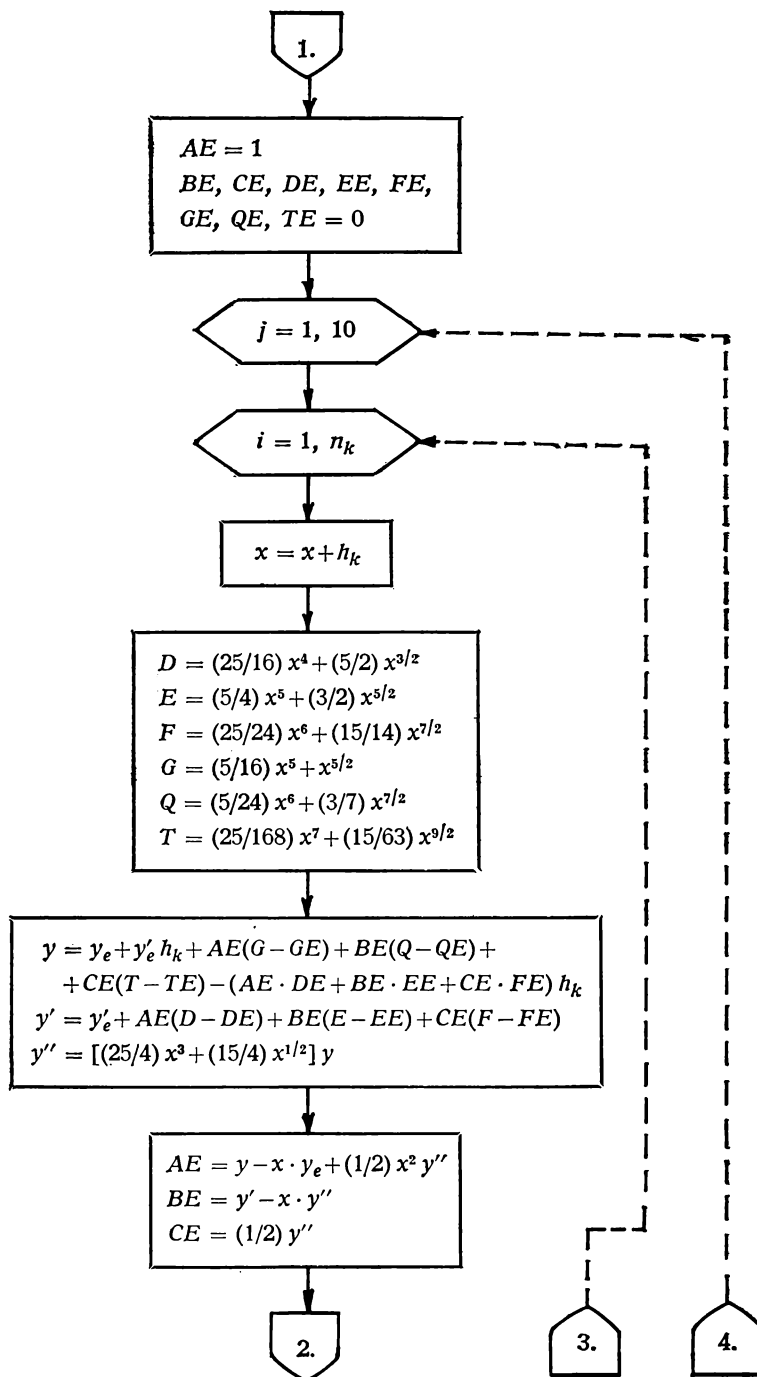
Program 1.

NUM. EXACT	RESULT RESULT	X=0.05	Y=	0.1011243063E 00 0.1011243073E 01	DY=	0.3391812347E 00 0.3391812381E 00
NUM. EXACT	RESULT RESULT	X=0.10	Y=	0.1032128068E 00 0.1032128089E 01	DY=	0.4895813300E 00 0.4895133988E 00
NUM. EXACT	RESULT RESULT	X=0.15	Y=	0.1059815376E 00 0.1059815409E 01	DY=	0.6156970954E 00 0.6156971142E 00
NUM. EXACT	RESULT RESULT	X=0.20	Y=	0.1093564645E 00 0.1093564691E 01	DY=	0.7335854654E 00 0.7335854962E 00
NUM. EXACT	RESULT RESULT	X=0.25	Y=	0.1133148391E 00 0.1133148453E 01	DY=	0.8498612936E 00 0.8498613399E 00
NUM. EXACT	RESULT RESULT	X=0.30	Y=	0.1178587514E 00 0.1178587594E 01	DY=	0.9683084511E 00 0.9683085169E 00
NUM. EXACT	RESULT RESULT	X=0.35	Y=	0.1230059706E 00 0.1230059808E 01	DY=	0.1091569704E 00 0.1091569794E 01
NUM. EXACT	RESULT RESULT	X=0.40	Y=	0.1287860243E 00 0.1287860369E 01	DY=	0.1221771503E 00 0.1221771623E 01
NUM. EXACT	RESULT RESULT	X=0.45	Y=	0.1352384136E 00 0.1352384292E 01	DY=	0.1360810287E 00 0.1360810444E 01
NUM. EXACT	RESULT RESULT	X=0.50	Y=	0.1424118829E 00 0.1424119020E 01	DY=	0.1510506122E 00 0.1510506324E 01
NUM. EXACT	RESULT RESULT	X=0.55	Y=	0.1503642898E 00 0.1503643130E 01	DY=	0.1672697128E 00 0.1672697386E 01
NUM. EXACT	RESULT RESULT	X=0.60	Y=	0.1591628690E 00 0.1591628970E 01	DY=	0.1849305423E 00 0.1849305748E 01
NUM. EXACT	RESULT RESULT	X=0.65	Y=	0.1688847855E 00 0.1688848192E 01	DY=	0.2042389006E 00 0.2042389412E 01
NUM. EXACT	RESULT RESULT	X=0.70	Y=	0.1796179293E 00 0.1796179695E 01	DY=	0.2254187121E 00 0.2254187626E 01
NUM. EXACT	RESULT RESULT	X=0.75	Y=	0.1914619295E 00 0.1914619775E 01	DY=	0.2487163420E 00 0.2487164044E 01
NUM. EXACT	RESULT RESULT	X=0.80	Y=	0.2045293852E 00 0.2045294424E 01	DY=	0.2744049651E 00 0.2744050417E 01
NUM. EXACT	RESULT RESULT	X=0.85	Y=	0.2189473202E 00 0.2189473880E 01	DY=	0.3027891826E 00 0.3027892764E 01
NUM. EXACT	RESULT RESULT	X=0.90	Y=	0.2348588770E 00 0.2348589573E 01	DY=	0.3342100408E 00 0.3342101551E 01
NUM. EXACT	RESULT RESULT	X=0.95	Y=	0.2524252765E 00 0.2524253715E 01	DY=	0.3690505883E 00 0.3690507272E 01
NUM. EXACT	RESULT RESULT	X=1.00	Y=	0.2718280700E 00 0.2718281822E 01	DY=	0.4077421046E 00 0.4077422729E 01

Result 1.

Flowchart 2.






```

0010 MASTER HEAD
0011 DIMENSION XZ(10),Z(10),Z1(10),Z2(10),N(2),H(2),ISTR(7,3)
0012 DATA ISTR(1,1)/28H EXACT VALUES /
0013 DATA ISTR(1,2)/28HNUMERICAL VALUES, H =0.01 /
0014 DATA ISTR(1,3)/28HNUMERICAL VALUES, H =0.001/
0015 H(1)=0.01
0016 H(2)=0.001
0017 N(1)=10
0018 N(2)=100
0019 CALL PRINT1
0020 CALL EXACT(XZ,Z,Z1,Z2)
0021 CALL PRINT2(XZ,Z,Z1,Z2,1,ISTR)
0022 DO 1 I=1,2
0023 CALL APPROX(XZ,Z1,Z2,H(I),N(I))
0024 1 CALL PRINT2(XZ,Z,Z1,Z2,I+1,ISTR)
0025 STOP
0026 END
0027 SUBROUTINE PRINT1
0028 WRITE(2,100)
0029 100 FORMAT(1H1//20X,62HAPPROXIMATE SOLUTION OF D. E.
0030 1(D**2-4.25*X**3-3.75*X**0.5)*Y=0/41X,19HBY SPLINE
2FUNCTIONS/)
0031 RETURN
0032 END
0033 SUBROUTINE PRINT2(XZ,Z,Z1,Z2,I,ISTRING)
0034 DIMENSION XZ(10),Z(10),Z1(10),Z2(10),ISTRING(7,3)
0035 WRITE(2,100)(ISTRING(J,I),J=1,7)
0036 100 FORMAT(///,30X,7A4/20X,61(1H-)/
0037 120X,7HI X 1,7X,4HY(X),6X,1HI,6X,5HY'(X),6X,1HI,
0038 26X,6HY''(X),5X,1HI/20X,61(1H-))
0039 DO 1 J=1,10
0040 1 WRITE(2,101) XZ(J),Z(J),Z1(J),Z2(J)
0041 101 FORMAT(20X,1HI,1X,F3.1,1X,1HI,3(1X,F15.12,1X,1HI))
0042 WRITE(2,102)
0043 102 FORMAT(20X,61(1H-))
0044 RETURN
0045 END
0046 SUBROUTINE APPROX(XZ,Z,Z1,Z2,H,N)
0047 DIMENSION XZ(10),Z(10),Z1(10),Z2(10)
0048 YE=1
0049 Y1E,Y2E,X=0
0050 AE=1
0051 BE,CE,DE,EE,FE,GE,QE,TE=0
0052 DO 2 J=1,10
0053 DO 1 I=1,N
0054 X=X+H
0055 D=(25./16.)*X**4+2.5*X**1.5
0056 E=1.25*X**5+1.5*X**2.5
0057 F=(25./24.)*X**6*(15./14.)*X**3.5
0058 G=(5./16.)*X**5+X**2.5
0059 Q=(5./24.)*X**6+(3./7.)*X**3.5
0060 T=(25./168.)*X**7+(15./63.)*X**4.5
0061 Y=YE+Y1E*H+AE*(G-GE)+BE*(Q-QE)+CE*(T-TE)
0062 1-(AE*DE+BE*EE+CE*FE)*H
0063 Y1=Y1E+AE*(D-DE)+BE*(E-EE)+CE*(F-FE)
0064 Y2=(6.25*X**3+3.75*SQRT(X))*Y
0065 AE=Y-X*Y1+0.5*X*X*Y2
0066 BE=Y1-X*Y2

```

```
0067         CE=0.5*Y2
0068         DE=D
0069         EE=E
0070         FE=F
0071         GE=G
0072         QE=Q
0073         TE=T
0074         YE=Y
0075         Y1E=Y1
0075         Y1E=Y1
0076     1  Y2E=Y2
0077         XZ(J)=X
0078         Z(J)=Y
0079         Z1(J)=Y1
0080         Z2(J)=Y2
0081     2  CONTINUE
0082         RETURN
0083         END
0084         SUBROUTINE EXACT(XZ,Z,Z1,Z2)
0085         DIMENSION XZ(10),Z(10),Z1(10),Z2(10)
0086         DO 1 I=1,10
0087             XZ(I)=I*0.1
0088             X=XZ(I)
0089             Z(I)=EXP(X**2.5)
0090             Z1(I)=2.5*Z(I)*X**1.5
0091     1  Z2(I)=Z(I)*(6.25*X**3+3.75*SQRT(X))
0092         RETURN
0093         END
```

Program 2.

APPROXIMATE SOLUTION OF D. E.
(D * * 2 - 4.25 * X * * 3 - 3.75 * X * * 0.5) * Y = 0
BY SPLINE FUNCTIONS

EXACT VALUES

X	Y(X)	Y' (X)	Y'' (X)
0.1	1.003167282935	0.079307337208	1.195879853621
0.2	1.018049502162	0.227642789125	1.758223393582
0.3	1.050530243140	0.431549333640	2.335023646708
0.4	1.106490046658	0.699805751159	3.066867585497
0.5	1.193364759448	1.054795233197	4.096701777248
0.6	1.321615432606	1.535578368202	5.623126754535
0.7	1.506762656208	2.206134146894	7.957552758975
0.8	1.772575196053	3.170878906850	11.617638577939
0.9	2.156385566864	4.602885687435	17.496507881904
1.0	2.718281828477	6.795704571179	27.182818284721

NUMERICAL VALUES, H=0.01

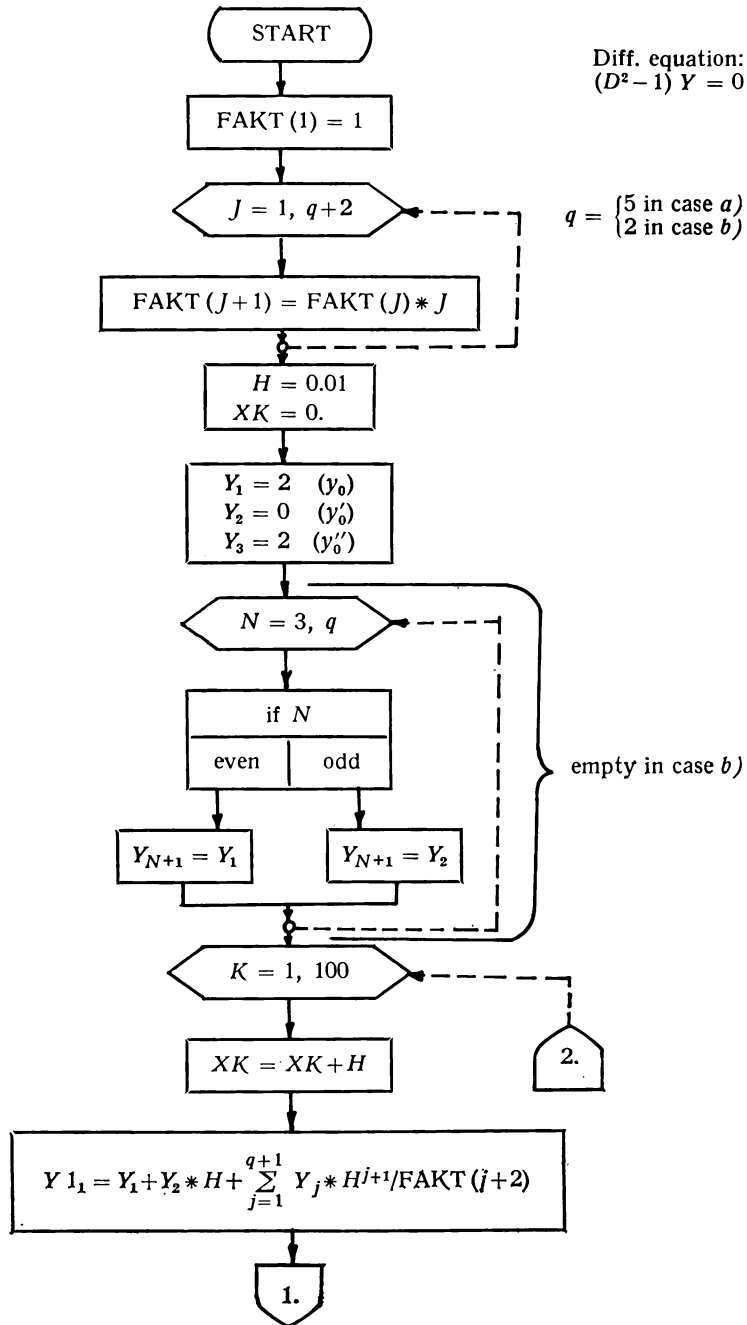
X	Y(X)	Y' (X)	Y'' (X)
0.1	1.003167280854	0.079307298794	1.195879851132
0.2	1.018049494625	0.227642715237	1.758223380573
0.3	1.050530226027	0.431549209628	2.335023608640
0.4	1.106490013582	0.699805544769	3.066867493674
0.5	1.193364519494	1.054794879717	4.096701571134
0.6	1.321615325708	1.535577739501	5.623126299293
0.7	1.506762464239	2.206132986233	7.957551744998
0.8	1.772574844115	3.170876689953	11.617636271517
0.9	2.156384903849	4.602881316327	17.496502503749
1.0	2.718280542117	6.795695703533	27.182805424317

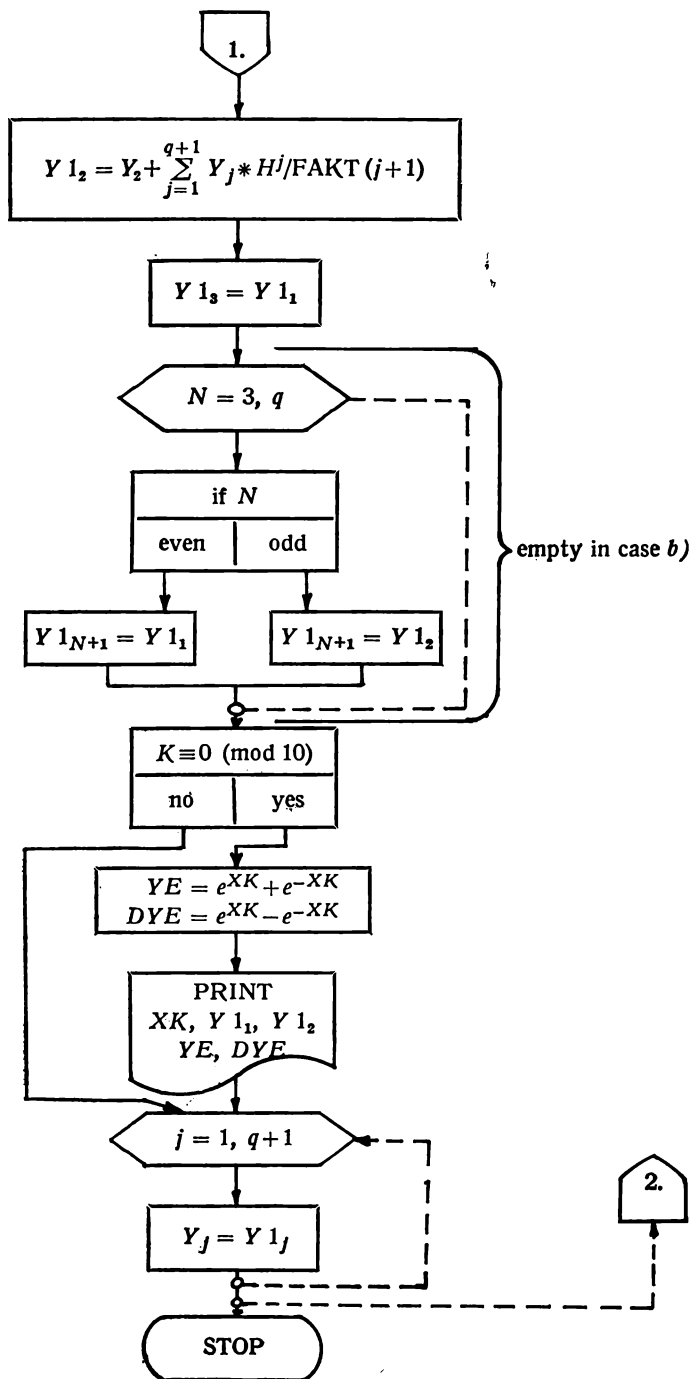
NUMERICAL VALUES, H=0.001

X	Y(X)	Y' (X)	Y'' (X)
0.1	1.003167282833	0.079307337187	1.195879853563
0.2	1.018049502133	0.227642789112	1.758223393698
0.3	1.050530243039	0.431549333622	2.335023646766
0.4	1.106490046571	0.699805751116	3.066867585729
0.5	1.193364579259	1.054795233154	4.096701777481
0.6	1.321615431661	1.535578366106	5.623126747317
0.7	1.506762654098	2.206134140578	7.957552737555
0.8	1.772575191861	3.170878892938	11.617638526949
0.9	2.156385559093	4.602885657281	17.496507771542
1.0	2.718281813780	6.795704515590	27.182818046767

Result 2.

Flowchart 3.





```

MASTER SPLINE
C**** APPROXIMATE SOLUTION OF THE DIFF. EQATION (D**2-1)*Y=0
C**** BY SPLINE FUNCTIONS TECHNIQUE.
REAL FAKT(8),Y(6).Y1(6)
WRITE (2,101)
101 FORMAT(1H1)
   FAKT(1)=1.
   DO 8 J=1,7
     8 FAKT(J+1)=FAKT(J)*J
   H=0.01
   XK=0.
   Y(1)=2
   Y(2)=0
   Y(3)=2
   DO 200 N=3,5
     IF (N-N/2*2.EQ.0) Y(N+1)=Y(1)
     IF (N-N/2*2.NE.0) Y(N+1)=Y(2)
200 CONTINUE
   DO 6 K=1,100
     XK=XK+H
     S=0.
     DO 201 J=1,6
       201 S=S+Y(J)*H**(J+1)/FAKT(J+2)
         Y1(1)=Y(1)+Y(2)*H+S
         S=0.
         DO 202 J=1,6
           202 S=S+Y(J)*H**J/FAKT(J+1)
             Y1(2)=Y(2)+S
             Y1(3)=Y(1)
             DO 203 N=3,5
               IF (N-N/2*2.EQ.0)Y1(N+1)=Y1(1)
               IF (N-N/2*2.NE.0)Y1(N+1)=Y1(2)
203 CONTINUE
             IF (K-K/10*10) 5,0,5
             YE=EXP(XK)+EXP(-XK)
             DYE=EXP(XK)-EXP(-XK)
             WRITE (2,100) XK,Y1(1),Y1(2),YE,DYE
100 FORMAT(1H0,12HNUM. RESULT,2X,2HX=,F4.2,2X,2HY=,D18.10,2X,
13HDY=,D18.10/13H EXACT RESULT,12X,D18.10,5X,D18.10)
     5 DO 6 J=1,6
       6 Y(J)=Y1(J)
     STOP
     END

```

NUM.	RESULT	X=0.10	Y=	0.2010008336E 01	DY=	0.2002335000E 00
EXACT	RESULT			0.2010008336E 01		0.2003335000E 00
NUM.	RESULT	X=0.20	Y=	0.2040133511E 01	DY=	0.4026720051E 00
EXACT	RESULT			0.2040133511E 01		0.4026720051E 00
NUM.	RESULT	X=0.30	Y=	0.2090677028E 01	DY=	0.6090405869E 00
EXACT	RESULT			0.2090677028E 01		0.6090405869E 00
NUM.	RESULT	X=0.40	Y=	0.2162144744E 01	DY=	0.8215046516E 00
EXACT	RESULT			0.2162144744E 01		0.8215046516E 00
NUM.	RESULT	X=0.50	Y=	0.2255251930E 01	DY=	0.1042190611E 01
EXACT	RESULT			0.2255251930E 01		0.1042190611E 01
NUM.	RESULT	X=0.60	Y=	0.2370930437E 01	DY=	0.1273307164E 01
EXACT	RESULT			0.2370930436E 01		0.1273307164E 01
NUM.	RESULT	X=0.70	Y=	0.2510338011E 01	DY=	0.1517167404E 01
EXACT	RESULT			0.2510338011E 01		0.1517167404E 01
NUM.	RESULT	Y=0.80	Y=	0.2674869893E 01	DY=	0.1776211964E 01
EXACT	R3ULT			0.2674869893E 01		0.1776211964E 01
NUM.	RESULT	X=0.90	Y=	0.2866172771E 01	DY=	0.2053033451E 01
EXACT	RESULT			0.2866172771E 01		0.2053033452E 01
NUM.	RESULT	X=1.00	Y=	0.3086161270E 01	DY=	0.2350402287E 01
EXACT	RESULT			0.3086161270E 01		0.2350402387E 01

Result 3./a

MASTER SPLINE

C**** APPROXIMATE SOLUTION OF THE DIFF. EQUATION $(D^{**2} - 1)*Y = 0$

C**** BY SPLINE FUNCTIONS TECHNIQUE.

REAL FAKT(5), Y(3), Y1(3)
WRITE (2,101)

101 FORMAT(1H1)

FAKT(1)=1.
DO 8 J=1,4

8 FAKT(J+1)=FAKT(J)*J

H=0.01
XK=0.
Y(1)=2
Y(2)=0
Y(3)=2
DO 6 K=1,100
XK=XK+H
S=0.
DO 201 J=1,3

201 S=S+Y(J)*H**(J+1)/FAKT(J+2)

Y1(1)=Y(1)+Y(2)*H+S
S=0.
DO 202 J=1,3

202 S=S+Y(J)*N**J/FAKT(J+1)

Y1(2)=Y(2)+S
Y1(3)=Y1(1)
IF(K-K/10*10) 5,0,5
YE=EXP(XK)+EXP(-XK)
DYE=EXP(XK)-EXP(-XK)
WRITE (2,100) XK, Y1(1), Y1(2), YE, DYE

100 FORMAT (1H0,12HNUM. RESULT,2X,2HX=,F4.2,2X,2HY=,D18.10,2X,
13HDY=,D18.10/13H EXACT RESULT,12X,D18.10,5X,D18.10)

5 DO 6 J=1,3

6 Y(J)=Y1(J)

STOP

END

Program 3./b

NUM.	RESULT	X=0.10	Y=	0.2010008336E 01	DY=	0.2003334996E 00
EXACT	RESULT			0.2010008336E 01		0.2003335000E 00
NUM.	RESULT	X=0.20	Y=	0.2040133511E 01	DY=	0.4026720035E 00
EXACT	RESULT			0.2040133511E 01		0.4026720051E 00
NUM.	RESULT	X=0.30	Y=	0.2090677028E 01	DY=	0.6090405832E 00
EXACT	RESULT			0.2090677028E 01		0.6090405869E 00
NUM.	RESULT	X=0.40	Y=	0.2162144743E 01	DY=	0.8215046449E 00
EXACT	RESULT			0.2162144744E 01		0.8215046516E 00
NUM.	RESULT	X=0.50	Y=	0.2255251929E 01	DY=	0.1042190600E 00
EXACT	RESULT			0.2255251930E 01		0.1042190611E 01
NUM.	RESULT	X=0.60	Y=	0.2370930434E 01	DY=	0.1273307149E 00
EXACT	RESULT			0.2370930436E 01		0.1273307164E 01
NUM.	RESULT	X=0.70	Y=	0.2510338006E 01	DY=	0.1517167382E 00
EXACT	RESULT			0.2510338011E 01		0.1517167404E 01
NUM.	RESULT	X=0.80	Y=	0.2674869885E 01	DY=	0.1776211935E 00
EXACT	RESULT			0.2674869893E 01		0.1776211964E 01
NUM.	RESULT	X=0.90	Y=	0.2866172760E 01	DY=	0.2053033413E 00
EXACT	RESULT			0.2866172771E 01		0.2053033452E 01
NUM.	RESULT	X=1.00	Y=	0.3086161255E 01	DY=	0.2350402339E 00
EXACT	RESULT			0.3086161270E 01		0.2350402387E 01

Result 3./b

Acknowledgement: The authors wish to thank Professor János Balázs for many helpful and stimulating discussions concerning this work. The authors are also very grateful to the Numerical and Computing Mathematics Department in The Eötvös Loránd University of Science in Budapest for achieving the programs in its computing centre.

REFERENCES

- [1] *János Balázs*. Private communications. Eötvös Loránd University of Science, Numerical Math. Dept., Budapest (Hungary).
- [2] *Tharwat Fawzy*. "Spline functions and the Cauchy problems, I." *Annales Univ. Sci. Budapest, Sectio Comp.*, 1, 1978, pp. 81–98.
- [3] *Tharwat Fawzy*. "Spline functions and the Cauchy problems, II." *Acta Math. Acad. Sci. Hungar.*, (to appear).
- [4] *Tharwat Fawzy*. "Spline functions and the Cauchy problems, III." *Annales Univ. Sci. Budapest, Sectio Comp.*, 1, 1978, pp. 35–45.
- [5] *Tharwat Fawzy*. "Spline functions and the Cauchy problems, IV." *Acta Math. Acad. Sci. Hungar.*, (to appear).

THARWAT FAWZY
Kuwait Institute of Applied Technology
Shuwaik – KUWAIT

ISTVÁN FEKETE
Department of Numerical Methods and Computer Science of the L. Eötvös University,
Budapest Hungary

JÁNOS KŐHEGYI
Department of Numerical Methods and Computer Science of the L. Eötvös University,
Budapest Hungary