# SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS, I.

## APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATION $y'' = f(x, y, y')$ WITH SPLINE FUNCTIONS

By

THARWAT FAWZY

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Abstract. A method for obtaining a spline function approximating the solution of a non-linear second order differential equation is presented. The existence, uniqueness and convergence of the approximate solution are investigated.

### 1. Introduction and description of the method

The necessity of accurate numerical approximations to the solution of non-linear ordinary differential equations governing physical systems has always been an important problem for scientists and engineers. The problem has been discussed and treated by many of mathematicians, with several methods, different degree of accuracy and different rates of convergence. The most effective methods, those are, the approximations by spline functions.

By spline functions, the Cauchy problem $y' = f(x, y)$ was discussed by F. R. Loscalzo and T. D. Talbot [3], [4]. The problem $y'' = f(x, y, y')$ was solved by K. D. Sharma and R. G. Gupta [7] by a one-step method based upon the Lobatto four-point quadrature formula in which the function $f$ is necessary to be sufficiently differentiable. The same problem was solved by Gh. Micula [5], [6] using the spline function, but only when the first derivative is absent i.e. $y'' = f(x, y)$ and for $f \in C^2$ at least. What is the situation if $f \in C^0$ and $C^1$? This question has not been treated till now by spline functions and that is our main task in this paper and the following.

In this paper we shall consider the Cauchy problem in the non-linear ordinary differential equation

$$(1.1) \qquad y''(x) = f[x, y(x), y'(x)] \qquad y(a) = y_0, \; y'(a) = y'_0$$

where $f \in C^0([a, b], R^2)$.

Our main purpose will be to study applications of spline functions to the numerical solution (1.1). We develop methods which produce smooth approximations to the solution $y$ in the form of piecewise polynomial function of degree $\leqq m$ which are joined at points called knots which are at least $k$ continuous derivatives, $k = m-3$. If $S$ is the spline function it satisfies

(1.2) $S \in C^k[a, b]$, where $k \leqq m-3$,

(1.3) $S \in \pi_m$ in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, (n-1)$.

Here $\pi_m$ denotes the set of all polynomials of degree $\leqq m$.
We define the knots by

(1.4) $$a = x_0 < x_1 < \ldots < x_n = b$$

and in our case we shall deal with equal subintervals and in this paper we denote

(1.5) $$h : x_{i+1} - x_i, \quad i = 0, 1, \ldots, (n-1).$$

Also in this paper and in what follows $c_0, c_1, c_2, \ldots$ will denote constants independent of $h$ and consequently independent of $n$.

We assume that (1.1) represents a single scalar equation, but nearly all of the numerical and theoretical considerations in this paper carry over to systems of second order equations where (1.1) could be treated in vector form. Moreover $f$ should satisfy sufficient conditions to guarantee that there exists a unique solution to (1.1). For most of the theoretical analysis we actually need to use the Lipschitz conditions on $f$.

Our method to approximate the solution of (1.1) will be divided into two main approximation processes, the first of which is to obtain, numerically, the approximate values $\bar{y}_i, \bar{y}_i', \bar{y}_i''$ for $i = 1, 2, \ldots, n$. The second approximation process is the smoothing of these approximate values by a suitable spline function, and thus, we get the required smooth approximate solution with sufficiently high degree of accuracy and high speed of convergence to the exact solution. Thus we start with the following items, in which we let the point be $a = O$ without loss of generality.

## 2. The first approximation process

This chapter contains some assumptions concerning the function $f$ and a method for obtaining the approximate values $\bar{y}_i$, $\bar{y}_i'$ and $\bar{y}_i''$ where $i = 1, 2, \ldots, n$ and also we shall discuss the convergence of these values to the exact ones.

**2.1** Assumptions and procedure of the method. In this paper we assume that $f(x, y, y')$ is a function on $R^3$ to $R$ defined and continuous in

$$D: |x - x_0| < \alpha, \quad |y - y_0| < \beta, \quad |y' - y_0'| < \gamma.$$

We also assume for all $(x, y, y')$, $(x, y_1, y_1')$, $(x, y_2, y_2')$ in $D$:

(2.1.1) $$|f(x, y, y')| \leqq M$$

and the Lipschitz condition

(2.1.2) $$|f(x,y_1,y_1')-f(x,y_2,y_2')|\leqq K(|y_1-y_2|+|y_1'-y_2'|)$$

where $K$ is the Lipschitz constant.

We assume also that $y''(x)\equiv f\{x,y(x),y'(x)\}$ has a modulus of continuity $w_0(f,h)=w_0(h)$.

Let $y(x)$ be the exact solution of (1.1) with the initial conditions $y(O)=$ $=y_0, y'(O)=y_0'$. Then by integrating (1.1) from $x_i$ to $x$ where $x_i\leqq x\leqq x_{i+1}$, $i=0,1,\dots,(n-1)$ we get

(2.1.3) $$y'(x)=y_i'+\int_{x_i}^{x} f\{t,y(t),y'(t)\}\,dt=$$

$$=y_i'+\int_{x_i}^{x} f\left\{t,y(t),y_i'+\int_{x_i}^{t} f[u,y(u),y'(u)]\,du\right\}dt$$

and

(2.1.4) $$y(x)=y_i+y_i'(x-x_i)+\int_{x_i}^{x}\int_{x_i}^{t} f\{u,y(u),y'(u)\}\,du\,dt$$

$$=y_i+y_i'(x-x_i)+\int_{x_i}^{x}\int_{x_i}^{t} f\{u,y(u),y_i'+\int_{x_i}^{u} f[v,y(v),y'(v)]\,dv\}\,du\,dt$$

and both of $y(x)$ and $y'(x)$ has the following Taylor expansion respectively where $x_i\leqq x\leqq x_{i+1}$, where $i=0,1,\dots,(n-1)$.

(2.1.5) $$y(x)=y_i+y_i'(x-x_i)+\frac{1}{2}y''(\xi_i)(x-x_i)^2 \quad x_i<\xi_i<x_{i+1}$$

and

(2.1.6) $$y'(x)=y_i'+y''(\eta_i)(x-x_i) \quad x_i<\eta_i<x_{i+1}$$

and these expansions may be approximated by using the approximate values $\bar{y}_i, \bar{y}_i'$ and $\bar{y}_i''$ (later will be defined) to get

(2.1.7) $$y_i^*(x)=\bar{y}_i+\bar{y}_i'(x-x_1)+\frac{1}{2}\bar{y}_i''(x-x_i)^2$$

and

(2.1.8) $$y_i^{*\prime}(x)=\bar{y}_i'+\bar{y}_i''(x-x_i)$$

Setting $x=x_{i+1}$ in (2.1.4) and (2.1.3) respectively we get

(2.1.9) $$y(x_{i+1})=y_{i+1}=y_i+y_i'h+\int_{x_i}^{x_{i+1}}\int_{x_i}^{t} f\{u,y(u),y'(u)\}\,du\,dt=$$

$$=y_i+y_i'h+\int_{x_i}^{x_{i+1}}\int_{x_i}^{t} f\{u,y(u),y_i'+\int_{x_i}^{u} f[v,y(v),y'(v)]\,dv\}\,du\,dt$$

and

$$(2.1.10)\qquad y'(x_{i+1}) = y'_{i+1} = y'_i + \int_{x_i}^{x_{i+1}} f[t, y(t), y'(t)]\,dt =$$

$$= y'_i + \int_{x_i}^{x_{i+1}} \left\{t, y(t), y'_i + \int_{x_i}^{t} f[u, y(u), y'(u)]\,du\right\} dt$$

and if we use the approximated Taylor expansions $y_i^*(x)$ and $y_i^{*\prime}(x)$ instead of $y(x)$ and $y'(x)$ in the integrands (2.1.9) and (2.1.10) and the approximate values $\bar{y}_i$ and $\bar{y}'_i$ instead of $y_i$ and $y'_i$ we get

$$(2.1.11)\qquad \bar{y}_{i+1} = \bar{y}_i + \bar{y}'_i h + \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f\{u, y_i^*(u), y_i^{**\prime}(u)\}\,du\,dt$$

and

$$(2.1.12)\qquad \bar{y}'_{i+1} = \bar{y}'_i + \int_{x_i}^{x_{i+1}} f\{t, y_i^*(t), y_i^{**\prime}(t)\}\,dt$$

where

$$y_{\cdot}^{**\prime}(x) = \bar{y}'_i + \int_{x_i}^{x} f[t, y_{\cdot}^*(t), y_i^{*\prime}(t)]\,dt$$

and after computing these approximate values $\bar{y}_{i+1}$ and $\bar{y}'_{i+1}$ we denote the approximate value $\bar{y}''_{i+1}$ to be

$$(2.1.13)\qquad \bar{y}''_{i+1} = f(x_{i+1}, \bar{y}_{i+1}, \bar{y}'_{i+1})$$

where in the above relations $i = 0, 1, \ldots, (n-1)$.

Before starting our calculations we can quitely use the substitutions $\bar{y}_0 = y_0$, $\bar{y}'_0 = y'_0$ and $\bar{y}''_0 = f(x_0, \bar{y}_0, \bar{y}'_0) = f(x_0, y_0, y'_0) = y''_0$ and so, for $x_0 \leq$ $\leq x \leq x_1$, we have the the following equations:

$$y(x) = y_0 + y'_0(x - x_0) + \frac{1}{2} y''(\xi_0)(x - x_0)^2 \qquad x_0 < \xi_0 < x_1$$

$$y'(x) = y'_0 + y''(\eta_0)(x - x_0) \qquad x_0 < \eta_0 < x_1$$

$$y_0^*(x) = y_0 + y'_0(x - x_0) + y''_0(x - x_0)^2$$

$$y_0^{*\prime}(x) = y'_0 + y''_0(x - x_0)$$

$$\bar{y}_1 = y_0 + y'_0 h + \int_{x_0}^{x_1} \int_{x_0}^{t} f[u, y_0^*(u), y_0^{**\prime}(u)]\,du\,dt$$

$$\bar{y}_1' = y_0' + \int_{x_0}^{x_1} f[t, y_0^*(t), y_0^{**\prime}(t)]\, dt$$

$$\bar{y}_1'' = f(x_1, \bar{y}_1, \bar{y}_1')$$

which will be used in proving a lemma given in the following paragraph.

**2.2** Convergence properties at $x = x_1$. In this paragraph we prove a lemma through which we can see how do the approxinate values converge to the exact values of $y(x)$, $y'(x)$ and $y''(x)$ at $x = x_1$ and in more details such that this technique in general case will be clear.

**Lemma 2.2** The following inequalities are true

$$|y_1' - \bar{y}_1'| \leqq c_0 w_0(f, h) h^3 \tag{2.2.1}$$

$$|y_1 - \bar{y}_1| \leqq c_1 w_0(f, h) h^4 \tag{2.2.2}$$

$$|y_1'' - \bar{y}_1''| \leqq c_2 w_0(f, h) h^3 \tag{2.2.3}$$

where, as it has been stated before, $c_0$, $c_1$ and $c_2$ are constants independent of $h$.

**Proof.** By using the last system of equations stated at the end of the paragraph 2.1 together with the Lipschitz condition on $f$ we get

$$|y_1' - \bar{y}_1'| = \left| y_0' + \int_{x_0}^{x_1} f[t, y(t), y'(t)]\, dt - \bar{y}_0' - \int_{x_0}^{x_1} f[t, y_0^*(t), y_0^{**\prime}(t)]\, dt \right|$$

$$\leqq \int_{x_0}^{x_1} |f[t, y(t), y'(t)] - f[t, y_0^*(t), y_0^{**\prime}(t)]|\, dt$$

$$\leqq K \int_{x_0}^{x_1} |y(t) - y_0^*(t)|\, dt + K \int_{x_0}^{x_1} |y'(t) - y_0^{**\prime}(t)|\, dt$$

$$= K \int_{x_0}^{x_1} \left| y_0 + y_0'(t - x_0) + \frac{1}{2} y''(\xi_0)(t - x_0)^2 - \bar{y}_0 - \bar{y}_0'(t - x_0) - \bar{y}_0''(t - x_0)^2 \right| dt +$$

$$+ K \int_{x_0}^{x_1} \left| y_0' + \int_{x_0}^{t} f[u, y(u), y'(u)]\, du - \bar{y}_0' - \int_{x_0}^{t} f[u, y_0^*(u), y_0^{*\prime}(u)]\, du \right| dt$$

$$\leqq K w_0(f, h) \frac{h^3}{6} + K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} |y(u) - y_0^*(u)|\, du dt +$$

$$+ K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} |y'(u) - y_0^{*\prime}(u)|\, du dt$$

$$= K\frac{h^3}{6} w_0(f, h) + K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} \Big| y_0 + y_0'(u-x_0) + \frac{1}{2} y''(\xi_0)(t-x_0)^2 -$$

$$- \bar{y}_0 - \bar{y}_0'(u-x_0) - \frac{1}{2} \bar{y}_0''(u-x_0)^2 \Big| \, dudt$$

$$+ K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} \left| y_0' + y''(\eta_0)(u-x_0) - \bar{y}_0' - \bar{y}_0''(u-x_0) \right| dudt$$

$$\leqq K\frac{h^3}{6} w_0(f, h) + K^2 \frac{h^4}{24} w_0(f, h) + K^2 \frac{h^3}{6} w_0(f, h)$$

$$\leqq c_0 w_0(f, h) h^3$$

and thus (2.2.1) is proved. Also,

$$|y_1 - \bar{y}_1| = \Big| y_0 + y_0' h + \int_{x_0}^{x_1} \int_{x_0}^{t} f[u, y(u), y'(u)] \, dudt - \bar{y}_0 - \bar{y}_0' h -$$

$$- \int_{x_0}^{x_1} \int_{x_0}^{t} f\{u, y_0^*(u), y_0^{**\prime}(u)\} \, dudt \Big|$$

$$\leqq \int_{x_0}^{x_1} \int_{x_0}^{t} |f\{u, y(u), y'(u)\} - f\{u, y_0^*(u), y_0^{**\prime}(u)\}| \, dudt$$

$$\leqq K \int_{x_0}^{x_1} \int_{x_0}^{t} |y(u) - y_0^*(u)| \, dudt + K \int_{x_0}^{x_1} \int_{x_0} |y'(u) - y_0^{**\prime}(u)| \, dudt$$

$$= K \int_{x_0}^{x_1} \int_{x_0}^{t} \Big| y_0 + y_0'(u-x_0) + \frac{1}{2} y_0''(\xi_0)(u-x_0)^2 - \bar{y}_0 - \bar{y}_0'(u-x_0) -$$

$$- \frac{1}{2} \bar{y}_0''(u-x_0)^2 \Big| \, dudt + K \int_{x_0}^{x_1} \int_{x_0}^{t} \Big| y_0' + \int_{x_0}^{u} f[v, y(v), y'(v)] \, dv - \bar{y}_0' -$$

$$- \int_{x_0}^{u} f[v, y_0^*(v), y_0^{*\prime}(v)] \, dv \Big| \, dudt$$

$$\leqq K\frac{h^4}{24} w_0(f, h) + K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} \int_{x_0}^{u} |y(v) - y_0^*(v)| \, dvdudt +$$

$$+ K^2 \int_{x_0}^{x_1} \int_{x_0}^{t} \int_{x_0}^{u} |y'(v) - y_0^{*\prime}(v)| \, dvdudt$$

$$= K\frac{h^4}{24} w_0(f,h) + K^2 \int_{x_0}^{x_1}\int_{x_0}^{t}\int_{x_0}^{u} \Big| y_0 + y_0'(v-x_0) + \frac{1}{2} y''(\xi_0)(v-x_0)^2 -$$

$$- \bar{y}_0 - \bar{y}_0'(v-x_0) - \frac{1}{2}\bar{y}_0''(v-x_0)^2 \Big| \, dvdudt + K^2 \int_{x_0}^{x_1}\int_{x_0}^{t}\int_{x_0}^{u} \cdot$$

$$\cdot \, |y_0' + y''(\eta_0)(v-x_0) - \bar{y}_0' - \bar{y}_0''(v-x_0)| \, dvdudt$$

$$\leqq K\frac{h^4}{24} w_0(f,h) + K\frac{h^5}{120} w_0(f,h) + K^2\frac{h^4}{24} w_0(f,h) \leqq c_1 w_0(f,h) h^4$$

and so, we have proved (2.2.2), and finally

$$|y_1'' - \bar{y}_1''| = |f[x_1, y_1, y_1'] - f[x_1, \bar{y}_1, \bar{y}_1']|$$
$$\leqq K|y_1 - \bar{y}_1| + K|y_1' - \bar{y}_1'|$$
$$\leqq Kc_1 w_0(f,h) h^4 + Kc_0 w_0(f,h) h^3$$
$$\leqq c_2 w_0(f,h) h^3$$

which is (2.2.3) and thus the proof of the lemma is complete.

**2.3** General convergence process. In this last paragraph of the second chapter we prove theorems dealing with the convergence of the approximate values $\bar{y}_{i+1}$, $\bar{y}_{i+1}'$ and $\bar{y}_{i+1}''$ to the exact values $y(x_{i+1})$, $y'(x_{i+1})$ and $y''(x_{i+1})$ where i in general may take the values $1, 2, \ldots, (n-1)$. Before proving these theorems we are in need to prove some lemmas.

**Lemma 2.3.1** The inequality

$$|y_{i+1}' - \bar{y}_{i+1}'| \leqq |y_i' - \bar{y}_i'|(1 + c_3 h) + Kh(1 + c_4 h)\,|y_i - \bar{y}_i| + c_5 w_0(f,h) h^3$$

is true and holds for all $i = 1, 2, \ldots, (n-1)$ where $K$ is the Lipschitz constant and $c_3$, $c_4$, $c_5$ are some other constants.

**Proof.** By the same way as in lemma 2.2 and by combination of equations (2.1.3), (2.1.5), (2.1.6), (2.1.7), (2.1.8), (2.1.10) and (2.1.12) together with the Lipschitz condition on $f$ it will be easy to deduce this inequality.

**Definition 2.3.1** We shall denote the estimating errors of $\bar{y}_i$ and of $\bar{y}_i'$ at any point $x_i \in [o, b]$, $i = 0, 1, \ldots, n$, to be as the following

$$e_i = |y_i - \bar{y}_i| \quad \text{and} \quad e_i' = |y_i' - \bar{y}_i'|$$

**Lemma 2.3.2** The inequality

$$e_{i+1}' \leqq c_6 e_{r_0} + c_7 w_0(h) h^2$$

is true for all

$$i = 0, 1, \ldots, (n-1), \quad \text{where} \quad e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$$

**Proof.** By using the definition 2.3.1 and lemma 2.3.1 the principle of successive substitution implies

$$
\begin{aligned}
e'_{i+1} &\leqq e'_i(1+c_3h) + Kh(1+c_4h)e_i + c_5w_0(h)h^3 \\
e'_i(1+c_3h) &\leqq e'_{i-1}(1+c_3h)^2 + Kh(1+c_4h)e_{i-1}(1+c_3h) + c_5w_0(h)h^3(1+c_3h) \\
e'_{i-1}(1+c_3h)^2 &\leqq e'_{i-2}(1+c_3h)^3 + Kh(1+c_4h)e_{i-2}(1+c_3h)^2 + c_5w_0(h)h^3(1+c_3h)^2 \\
&\leqq \qquad + \qquad + \\
&\leqq \qquad + \qquad + \\
&\leqq \qquad + \qquad + \\
e'_1(1+c_3h)^i &\leqq e'_0(1+c_3h)^{i+1} + Kh(1+c_4h)e_0(1+c_3h)^i + c_5w_0(h)h^3(1+c_3h)^i
\end{aligned}
$$

and easily we obtain

$$e'_{i+1} \leqq e'_0(1+c_3h)^{i+1} + Kh(1+c_4h)\sum_{j=0}^{i} e_j(1+c_3h)^{i-j} + $$

$$+ c_5w_0(h)h^3 \sum_{j=0}^{i}(1+c_3h)^j .$$

Let $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$, $0 \leqq r_0 \leqq i$, and substitute $e'_0$ by zero to obtain

$$e'_{i+1} \leqq Kh(1+c_4h)e_{r_0}\sum_{j=0}^{i}(1+c_3h)^j + c_5w_0(h)h^3\sum_{j=0}^{i}(1+c_3h)^j$$

$$= Kh(1+c_4h)e_{r_0}\frac{\{(1-c_3h)^{i+1}-1\}}{c_3h} + c_5w_0(h)h^3\frac{\{(1+c_3h)^{i+1}-1\}}{c_3h}$$

and

$$(1+c_3h)^{i+1} = \left(1+c_3\frac{b}{n}\right)^{i+1} \leqq \left(1+c_3\cdot\frac{b}{n}\right)^n \leqq e^{bc_3} = \text{constant}$$

implies

$$e'_{i+1} \leqq c_6e_{r_0} + c_7w_0(h)h^2$$

which completes the proof.

**Lemma 2.3.3** The inequality

$$e_{i+1} \leqq e_i(1+c_8h^2) + c_9he'_i + c_{10}w_0(h)h^4$$

is true for all $i = 0, 1, \ldots, (n-1)$.

**Proof.** By the same way as in lemma 2.3.1 and by using the equations (2.1.9), (2.1.11), (2.1.5), (2.1.6), (2.1.7), (2.1.8) and the Lipschitz condition (2.1.2) we can get the required result.

**Lemma 2.3.4** The inequality

$$e_{i+1} \leqq e_{r_0}(1+c_{11}h) + c_{12}w_0(h)h^3$$

is true for all $i = 0, 1, \ldots, (n-1)$, where $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ and $0 \leqq r_0 \leqq i$.

**Proof.** From the lemma 2.3.2 we get

$$e_i' \leqq c_6\, e_{r_0}^* + c_7\, w_0(h)\, h^2$$

where $e_{r_0}^* = \max\{e_0, e_1, \ldots, e_{i-1}\}$ and by recalling $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ then obviously $e_{r_0}^* \leqq e_{r_0}$ from which we get

$$e_i' \leqq c_6\, c_{r_0} + c_7\, w_0(h)\, h^2$$

Using this result in the lemma 2.3.3 we get

$$e_{i+1} \leqq e_{r_0}(1 + c_8 h^2) + c_9 h\{c_6\, e_{r_0} + c_7\, w_0(h)\, h^2\} + c_{10}\, w_0(h)\, h^4$$

i.e.

$$e_{i+1} \leqq e_{r_0}(1 + c_{11} h) + c_{12}\, w_0(h)\, h^3$$

and thus the proof is complete.

**Lemma 2.3.5** The inequality

$$e_{r_0}' \leqq c_{13}(1 + c_{14} h)\, e_{r_1} + c_{15}\, w_0(h)\, h^2$$

is true where $r_0$ is the subscript of the maximum error $e_{r_0}$ where $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ and $e_{r_1} = \max\{e_0, e_1, \ldots, e_{r_0-1}\}$ with some $r_1$, $0 \leqq r_1 \leqq r_0 - 1$.

**Proof.** By using the lemma 2.3.1 but the interval $[x_i, x_{i+1}]$ is replaced by the interval $[x_{r_0-1}, x_{r_0}]$ and by similar procedures as shown in lemma 2.3.2 with the using of the definition 2.3.1 it would be easy to obtain the required result.

**Lemma 2.3.6** The inquality

$$e_{r_0} \leqq e_{r_1}(1 + c_{16} h) + c_{17}\, w_0(h)\, h^3$$

is true where $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ with some $r_0$, $0 \leqq r_0 \leqq i$, and $e_{r_1} = \max\{e_0, e_1, \ldots, e_{r_0-1}\}$ with some $r_1$, $0 \leqq r_1 \leqq r_0 - 1$.

**Proof.** From lemma 2.3.3 with replacing the interval $[x_i, x_{i+1}]$ by the interval $[x_{r_0-1}, x_{r_0}]$ we obviously get

$$e_{r_0} \leqq e_{r_0-1}(1 + c_{18} h^2) + c_{19}\, h e_{r_0-1}' + c_{20}\, w_0(h)\, h^4$$

and $e_{r_1} = \max\{e_0, e_1, \ldots, e_{r_0-1}\}$ implies

$$e_{r_0-1} \leqq e_{r_1}$$

from which we get

$$e_{r_0} \leqq e_{r_1}(1 + c_{18} h^2) + c_{19}\, h e_{r_0-1}' + c_{20}\, w_0(h)\, h^4$$

From lemma 2.3.5 we get

$$e_{r_0-1}' \leqq c_{13}(1 + c_{14} h)\, e_{r_1*} + c_{15}\, w_0(h)\, h^2$$

where $e_{r_1*} = \max\{e_0, e_1, \ldots, e_{r_0-2}\}$ for some $r_1^*$, $0 \leqq r_1^* \leqq r_0 - 2$, and it is easy to use the fact that

$$e_{r_1^*} = \max\{e_0, \ldots, e_{r_0-2}\} \leqq \max\{e_0, e_1, \ldots, e_{r_0-2}, e_{r_0-1}\} = e_{r_1}$$

from which we get

$$e'_{r_0-1} \leqq c_{13}(1+c_{14}h)\,e_{r_1} + c_{15}\,w_0(h)\,h^2$$

Returning to $e_{r_0}$ and using the last inequality we get

$$e_{r_0} \leqq e_{r_1}(1+c_{18}h^2) + c_{19}h\{c_{13}(1+c_{14}h)\,e_{r_1} + c_{15}\,w_0(h)\,h^2\} + c_{20}\,w_0(h)\,h^4$$
$$\leqq e_{r_1}(1+c_{16}h) + c_{17}\,w_0(h)\,h^3$$

which completes the proof.

**Theorem 2.3.1** The speed of convergence of the approximate value $\bar{y}_{i+1}$, given by the formula (2.1.11), to the exact value of the solution of (1.1) at $x_{i+1}$ is estimated by the inequality

$$e_{i+1} = |y_{i+1} - \bar{y}_{i+1}| \leqq c_{21}\,w_0(h)\,h^2$$

which holds for all $i = 0, 1, \ldots, (n-1)$.

**Proof.** From lemma 2.3.4 we have

$$e_{i+1} \leqq e_{r_0}(1+c_{11}h) + c_{12}\,w_0(h)\,h^3$$

where $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ with some $r_0, 0 \leqq r_0 \leqq i$, and $i = 0, 1, \ldots, (n-1)$

And from lemma 2.3.6 we know that

$$e_{r_0} \leqq e_{r_1}(1+c_{16}h) + c_{17}\,w_0(h)\,h^3$$

where $e_{r_1} = \max\{e_0, e_1, \ldots, e_{r_0-1}\}$ with some $r_1$, $0 \leqq r_1 \leqq r_0 - 1$.

Continuing by the same way as it was shown in lemma 2.3.4 and lemma 2.3.6 we can obtain the following inequalities

$$e_{r_1} \leqq e_{r_2}(1+c_1^*h) + c_1^{**}\,w_0(h)\,h^3$$

where $e_{r_2} = \max\{e_0, e_1, \ldots, e_{r_1-1}\}$ with some $r_2$, $0 \leqq r_2 \leqq r_1 - 1$, and

$$e_{r_2} \leqq e_{r_3}(1+c_2^*h) + c_2^{**}\,w_0(h)\,h^3$$

where $e_{r_3} = \max\{e_0, e_1, \ldots, e_{r_2-1}\}$ with some $r_3$, $0 \leqq r_3 \leqq r_2 - 1$, and at the end we can get the inequality

$$e_{r_s} \leqq e_{r_{s+1}}(1+c_s^*h) + c_s^{**}\,w_0(h)\,h^3$$

where $e_{r_s} = \max\{e_0, e_1\}$ with some $r_s$, $0 \leqq r_s \leqq 1$, and $e_{r_{s+1}} = \max\{e_0\} = e_0$ for some $r_{r_{s+1}}$, $0 \leqq r_{s+1} \leqq 0$, i.e. $r_{s+1} = 0$.

Now, taking $c_{22} = \max\{c_{11}, c_{16}, c_1^*, c_2^*, \ldots, c_s^*\}$ and $c_{20} = \max\{c_{12}, c_{17}, c_1^{**}, c_2^{**}, \ldots, c_s^{**}\}$ and by the rearrangement of the above inequalities we get

$$
\begin{array}{rlll}
e_{i+1} & \leqq e_{r_0}(1+c_{22}h) & & + c_{23} w_0(h) h^3 \\
e_{r_0}(1+c_{22}h) & \leqq e_{r_1}(1+c_{22}h)^2 & & + c_{23} w_0(h) h^3 (1+c_{22}h) \\
e_{r_1}(1+c_{22}h)^2 & \leqq e_{r_2}(1+c_{22}h)^3 & & + c_{23} w_0(h) h^3 (1+c_{22}h)^2 \\
& \leqq \quad \cdot & & + \quad \cdot \\
& \leqq & & + \\
& \leqq & & + \quad \cdot \\
e_{r_s}(1+c_{22}h)^{s+1} & \leqq e_{r_{s+1}}(1+c_{22}h)^{s+2} & & + c_{23} w_0(h) h^3 (1+c_{22}h)^{s+1}
\end{array}
$$

from which we get

$$e_{i+1} \leqq e_{r_{s+1}}(1+c_{22}h)^{s+2} + c_{23} w_0(h) h^3 \sum_{j=0}^{s+1} (1+c_{22}h)^j$$

and using the fact that $e_{r_{s+1}} = e_0 = 0$ this will be

$$
\begin{aligned}
& \leqq c_{23} w_0(h) h^3 \sum_{j=0}^{s+1} (1+c_{22}h)^j \\
& = c_{23} w_0(h) h^3 \frac{\{(1+c_{22}h)^j - 1\}}{c_{22}h} \\
& \leqq c_{21} w_0(h) h^2
\end{aligned}
$$

and this completes the proof.

**Theorem 2.3.2** The speed of convergence of the approximate value $\bar{y}'_{i+1}$, given by the formula (2.1.12), to $y'_{i+1}$ is estimated by the inequality

$$e'_{+1} = |y'_{i+1} - \bar{y}'_{i+1}| \leqq c_{24} w_0(h) h^2$$

where $i = 0, 1, \ldots, (n-1)$.

**Proof.** The lemma 2.3.2 tells us that

$$e'_{i+1} \leqq c_6 e_{r_0} + c_7 w_0(h) h^2$$

where $e_{r_0} = \max\{e_0, e_1, \ldots, e_i\}$ and from theorem 2.3.1 we can obtain that

$$e_{r_0} \leqq c_{21} w_0(h) h^2$$

and thus we obviously get

$$
\begin{aligned}
e'_{i+1} & \leqq c_6 c_{21} w_0(h) h^2 + c_7 w_0(h) h^2 \\
& \leqq c_{24} w_0(h) h^2
\end{aligned}
$$

which completes the proof.

**Theorem 2.3.3** The error of the approximate value $\bar{y}''_{i+1}$ is estimated by the inequality

$$e_{i+1} = |y''_{i+1} - \bar{y}''_{i+1}| \leqq c_{25}\, w_0(h)\, h^2$$

where $i = 0, 1, \ldots, (n-1)$.

**Proof.** Using equations (1.1) and (2.1.13) we get

$$|y''_{i+1} - \bar{y}''_{i+1}| = |f(x_{i+1}, y_{i+1}, y'_{i+1}) - f(x_{i+1}, \bar{y}_{i+1}, \bar{y}'_{i+1})|$$

applying the Lipschitz condition on $f$ this will be

$$\leqq K(|y_{i+1} - \bar{y}_{i+1}| + |y'_{i+1} - \bar{y}'_{i+1}|)$$

using theorems 2.3.1 and 2.3.2 this reduces to

$$\leqq K(c_{21}\, w_0(h)\, h^2 + c_{24}\, w_0(h)\, h^2)$$
$$\leqq c_{25}\, w_0(h)\, h^2$$

which completes the proof.

## 3. The second approximation process

In the last chapter we have obtained a set of points $\bar{Y}: \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n$ which are the approximate values of the exact solution $y(x)$ of (1.1) at the points $x_0, x_1, \ldots, x_n$ respectively. Also we obtained two sets of approximate values $\bar{Y}': \bar{y}'_0, \bar{y}'_1, \ldots, \bar{y}'_n$ and $\bar{Y}'': \bar{y}''_0, \bar{y}''_1, \ldots, \bar{y}''_n$ of the values $y'(x_i)$ respectively where $i = 0, 1, 2, \ldots, n$.

Here in this chapter and on the bases of those sets of approximate values $\bar{Y}$, $\bar{Y}'$ and $\bar{Y}''$ we are going to construct a spline function $S_\Delta(x)$ which will be interpolated to the set $\bar{Y}$ on the mesh $\Delta$ and approximates the exact solution $y(x)$ of (1.1) and also we shall discuss the convergence of this function to $y(x)$.

**3.1** Construction of the spline function. In this paragraph we shall introduce the spline function approximating the solution of our differential equation and so we prove the following theorem.

**Theorem 3.1** For a given mesh of points

$$\Delta: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b\,, \quad x_{k+1} - x_k = h$$

and given sets of points

$$\bar{Y}: \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_k, \bar{y}_{k+1}, \ldots, \bar{y}_n$$

and

$$\bar{Y}': \bar{y}'_0, \bar{y}'_1, \ldots, \bar{y}'_k, \bar{y}'_{k+1}, \ldots, \bar{y}'_n$$

and

$$\bar{Y}'': \bar{y}''_0, \bar{y}''_1, \ldots, \bar{y}''_k, \bar{y}''_{k+1}, \ldots, \bar{y}''_n$$

there is a unique spline function $S_\Delta(x)$ interpolated on the mesh $\Delta$ to the set $\bar{Y}$ and satisfies the following conditions

(3.1.1) $$S_\Delta(\overline{Y}, x) = S_\Delta(x) \in C^2[0, b]$$

(3.1.2) $$S_k(x_k) = \bar{y}_k \quad (k = 0, 1, \ldots, n)$$

(3.1.3) $$S'_k(x_k) = \bar{y}'_k \quad (k = 0, 1, \ldots, n)$$

(3.1.4) $$S''_k(x_k) = \bar{y}''_k \quad (k = 0, 1, \ldots, n)$$

For $x_k \leqq x \leqq x_{k+1}$ and $k = 0, 1, \ldots, (n-1)$

$$S_\Delta(x) = S_k(x) = \bar{y}_k - \bar{y}'_k(x-x_k) + \frac{1}{2}\bar{y}''_k(x-x_k)^2 + a_1^k(x-x_k)^3 +$$

(3.1.5) $$+ a_2^k(x-x_k)^4 + a_3^k(x-x_k)^5$$

**Proof.** From the continuity condition (3.1.1) and for $x = x_{k+1}$ we get

(3.1.6) $$S_k(x_{k+1}) = S_{k+1}(x_{k+1}) = \bar{y}_{k+1}, \quad k = 0, 1, 2, \ldots, (n-1).$$

(3.1.7) $$S'_k(x_{k+1}) = S'_{k+1}(x_{k+1}) = \bar{y}'_{k+1}, \quad k = 0, 1, 2, \ldots, (n-1).$$

(3.1.8) $$S'_k(x_{k+1}) = S''_{k+1}(x_{k+1}) = \bar{y}''_{k+1}, \quad k = 0, 1, 2, \ldots, (n-1).$$

the above three equations are implied also by using the conditions (3.1.1), (3.1.2), (3.1.3) and (3.1.4). Using the equation (3.1.5) the equations (3.1.6), (3.1.7), and (3.1.8) imply the following

(3.1.9) $$a_1^k + a_2^k h + a_3^k h^2 = F_k,$$

$$F_k = \frac{1}{h^3}\left(\bar{y}_{k+1} - \bar{y}_k - \bar{y}'_k h - \frac{1}{2}\bar{y}''_k h^2\right),$$

and

(3.1.10) $$3a_1^k + 4a_2^k h + 5a_3^k h^2 = F'_k,$$

$$F'_k = \frac{1}{h^2}(\bar{y}'_{k+1} - \bar{y}'_k - y''_k h),$$

and

(3.1.11) $$6a_1^k + 12a_2^k h + 20a_3^k h^2 = F''_k,$$

$$F''_k = \frac{1}{h}(\bar{y}''_{k+1} - \bar{y}''_k),$$

where $h = x_{k+1} - x_k > 0$ and $k = 0, 1, 2, \ldots, (n-1)$.

These last three equations in the three unknowns $a_1^k$, $a_2^k$ and $a_3^k$ have a unique solution since its determinant is different from zero for $h > 0$ and the solutions are

(3.1.12) $$a_1^k = \frac{1}{2}(20F_k - 8F'_k + F''_k)$$

(3.1.13) $$a_2^k = \frac{1}{2h}(-30F_k + 14F_k' - 2F_k'')$$

(3.1.14) $$a_3^k = \frac{1}{2h^2}(12F_k - 6F_k' + F_k'').$$

The uniqueness of this solution guarantees the uniqueness of the spline function $S_\Delta(x)$ and consequently the existance of such a function and thus the theorem is proved.

**3.2** Convergence of the spline function to the solution. In this paragraph we prove the essential theorem concerned with the convergence of our spline function constructed in theorem 3.1 to the exact solution of the differential equation in consideration.

**Theorem 3.2.1** If $y(x)$ is the solution of (1.1) and $S_\Delta(x)$ is the spline function represented in theorem 3.1, then there exists a constant $E$ independent of $h$ such that

$$|y^{(q)}(x) - S^{(q)}(x)| \leqq E w_0(f, h)\, h^{(2-q)} \quad (q = 0, 1, 2.)$$

for all $x \in [0, b]$.

For the proof of this theorem we are in need to the following lemma.

**Lemma 3.2.1** The following inequalities are true

$$|a_j^k| \leqq \frac{A_j}{h^j} w_0(f, h) \quad (j = 1, 2, 3.)$$

where $A_j$ $(j = 1, 2, 3)$ are constants independent of $h$.

**Proof of the lemma.** For the proof of this lemma we deduce at first some inequalities concerning the absolute values of $F_k$, $F_k'$ and $F_k''$. They are calculated as follows.

From (3.1.9) we have

$$|F_k| = \frac{1}{h^3}\left|\bar{y}_{k+1} - \bar{y}_k - \bar{y}_k' h - \frac{1}{2}\bar{y}_k'' h^2\right|$$

and from Taylor expansion of $y(x)$ and for $x = x_{k+1}$ this will be

$$= \frac{1}{h^3}\left|\bar{y}_{k+1} - \bar{y}_k - \bar{y}_k' h - \frac{1}{2}\bar{y}_k'' h^2 - y_{k+1} + y_k + y_k' h + \frac{1}{2} y''(\xi_k) h^2\right|$$

$$\leqq \frac{1}{h^3}\left\{|y_{k+1} - \bar{y}_{k+1}| + |y_k - \bar{y}_k| + h\,|y_k' - \bar{y}_k'| + \frac{1}{2}h^2\,|y''(\xi_k) - \bar{y}_k''|\right\}$$

and by using theorems 2.3.1, 2.3.2 and 2.3.3 this will be

$$\leqq \frac{B}{h} w_0(f, h)$$

where $B$ is a constant independent of $h$.

The same will be done for $F_k'$, and from (3.1.10) we have

$$|F_k'| = \frac{1}{h^2} |\bar{y}_{k+1}' - \bar{y}_k' - \bar{y}_k'' h|$$

and using the Taylor expansion of $y'(x)$ and for $x = x_{k+1}$, this equals

$$= \frac{1}{h^2} |y_{k+1}' - \bar{y}_k' - \bar{y}_k'' h - y_{k+1}' + y_k' + y''(\eta_k) h|$$

$$\leqq \frac{1}{h^2} \{|y_{k+1}' - \bar{y}_{k+1}'| + |y_k' - \bar{y}_k'| + h |y''(\eta_k) - \bar{y}_k''|\}.$$

Using theorems 2.3.2, 2.3.3 and the modulus of continuity of $f$, this becomes

$$\leqq \frac{B'}{h} w_0(f, h)$$

where $B'$ is some constant independent of $h$.

For $|F_k''|$ we have from (3.1.11)

$$|F_k''| \leqq \frac{1}{h} |\bar{y}_{k+1}'' - \bar{y}_k''|$$

$$\leqq \frac{1}{h} |\bar{y}_{k+1}'' - \bar{y}_k'' - (y_{k+1}'' - y_k'') + w_0(f, h)|$$

$$\leqq \frac{1}{h} (|y_{k+1}'' - \bar{y}_{k+1}''| + |y_k'' - \bar{y}_k''| + w_0(f, h))$$

$$\leqq \frac{B''}{h} w_0(f, h)$$

where $B''$ is a constant independent of $h$.

Taking the last three inequalities concerning $|F_k|$, $|F_k'|$ and $|F_k''|$ into consideration, and using equations (3.1.12), (3.1.13) and (3.1.14) the proof of this lemma will be complete.

**Proof of theorem 3.2.1** We start the proof with the case $q = 0$. For this case and for $x_k \leqq x \leqq x_{k+1}$ we have by (2.1.5) and (3.1.5) the following

$$|y(x) - S_\Delta(x)| = \Big| y_k + y_k'(x - x_k) + \frac{1}{2} y''(\xi_k)(x - x_k)^2 - \bar{y}_k - \bar{y}_k'(x - x_k) -$$

$$- \frac{1}{2} \bar{y}_k''(x - x_k)^2 - a_1^k (x - x_k)^3 - a_2^k (x - x_k)^4 - a_3^k (x - x_k)^5 \Big|$$

$$\leqq |y_k - \bar{y}_k| + h |y_k' - \bar{y}_k'| + \frac{1}{2} h^2 |y''(\xi_k) - \bar{y}_k''| + h^3 |a_1^k| +$$

$$+ h^4 |a_2^k| + h^5 |a_3^k| .$$

Applying theorems 2.3.1, 2.3.2, 2.3.3 and the lemma 3.2.1 this will be

$$\leqq c_{21} w_0(f,h) h^2 + c_{24} w_0(f,h) h^3 + \frac{1}{2} w_0(f,h) h^2 + \frac{1}{2} c_{25} w_0(h) h^4 +$$

$$+ A_1 w_0(f,h) h^2 + A_2 w_0(f,h) h^2 + A_3 w_0(f,h) h^2$$

$$= \left(c_{21} + c_{24} h + \frac{1}{2} + \frac{1}{2} c_{25} h^2 + A_1 + A_2 + A_3\right) w_0(f,h) h^2$$

$$\leqq E_0 w_0(f,h) h^2 ,$$

where $E_0$ is a constant independent of $h$.

Secondly, the case when $q = 1$. By the same technique as the above but using (2.1.6) and the derivative of (3.1.5) instead of (2.1.5) and (3.1.5) we get,

$$|y'(x) - S'_\Delta(x)| = |y'_k + y''(\eta_k)(x-x_k) - \bar{y}'_k - \bar{y}''_k(x-x_k) -$$

$$- 3a_1^k (x-x_k)^2 - 4a_2^k (x-x_k)^3 - 5a_3^k (x-x_k)^4|$$

$$\leqq |y'_k - \bar{y}'_k| + h\,|y''(\eta_k) - \bar{y}''_k| + 3\,|a_1^k|\, h^2 + 4\,|a_2^k|\, h^3 + 5\,|a_3^k|\, h^4 .$$

Applying theorem 2.3.2, 2.3.3 and lemma 3.2.1 this becomes

$$\leqq c_{24} w_0(f,h) h^2 + w_0(f,h) h + 3A_1 w_0(f,h) h + c_{25} w_0(h) h^3 +$$

$$+ 4A_2 w_0(f,h) h + 5A_3 w_0(f,h) h$$

$$= (c_{24} h + 1 + 3A_1 + c_{25} h^2 + 4A_2 + 5A_3) w_0(f,h) h$$

$$\leqq E_1 w_0(f,h) h$$

where $E_1$ is a constant independent of $h$.

At last, the case when $q = 2$. In this case and using the second derivative of (3.1.5) we get

$$|y''(x) - S''_\Delta(x)| = |y''(x) - \bar{y}''_k - 6a_1^k (x-x_k) - 12a_2^k (x-x_k)^2 - 20a_3^k (x-x_k)^3|$$

$$\leqq |y''(x) - \bar{y}''_k| + 6\,|a_1^k|\, h + 12\,|a_2^k|\, h^2 + 20\,|a_3^k|\, h^3$$

$$= |y''(x) - \bar{y}''_k - y''_k + y''_k| + 6\,|a_1^k|\, h + 12\,|a_2^k|\, h^2 + 20\,|a_3^k|\, h^3$$

$$\leqq |y''(x) - y''_k| + |y''_k - \bar{y}''_k| + 6\,|a_1^k|\, h +$$

$$+ 12\,|a_2^k|\, h^2 + 20\,|a_3^k|\, h^3 .$$

Using theorem 2.3.2 and lemma 3.2.1 this becomes

$$\leqq w_0(f,h) + c_{25} w_0(f,h) h^2 + 6A_1 w_0(f,h) + 12A_2 w_0(f,h) + 20A_3 w_0(f,h)$$

$$= (1 + c_{25} h^2 + 6A_1 + 12A_2 + 20A_3) w_0(f,h) \leqq E_2 w_0(f,h)$$

where $E_2$ is some constant independent of $h$.

Now, by taking $E = \max(E_0, E_1, E_2)$, the proof of the theorem is complete.

In the following theorem we shall prove that our approximate solution $S_\Delta(x)$ satisfies the differential equation (1.1) as $n \to \infty$ or as $h \to 0$.

**Theorem 3.2.2** If $\bar{S}''_\Delta(x)$ denotes the function

$$\bar{S}''_\Delta(x) = f[x, S_\Delta(x), S'_\Delta(x)]$$

and $S_\Delta(x)$ is the spline function given in theorem 3.1, then for any $x \in [0, b]$

$$|\bar{S}''_\Delta(x) - S''_\Delta(x)| \leqq M w_0(f, h)$$

where $M$ is some constant independent of $h$. Otherwise

$$S''_\Delta(x) \cong \bar{S}''_\Delta(x) \quad \text{as} \quad n \to \infty \quad \text{or as} \quad h \to 0 .$$

**Proof.** We have

$$\begin{aligned} |\bar{S}''_\Delta(x) - S''_\Delta(x)| &= |\bar{S}''_\Delta(x) - y''(x) + y''(x) - S''_\Delta(x)| \\ &\leqq |\bar{S}''_\Delta(x) - y''(x)| + |y''(x) - S''_\Delta(x)| \end{aligned}$$

using the definition of $\bar{S}''_\Delta(x)$ this will be

$$= |f[x, S_\Delta(x), S'_\Delta(x) - f[x, y(x), y'(x)]| + |y''(x) - S''_\Delta(x)|$$

applying the Lipschitz condition on $f$ this will be

$$\leqq K(|S_\Delta(x) - y(x)| + |S'_\Delta(x) - y'(x)|) + |y''(x) - S''(x)|$$

and by using theorem 3.2.1 this reduces to

$$\begin{aligned} &\leqq KEw_0(f, h)h^2 + KEw_0(f, h)h + Ew_0(f, h) \\ &= (KEh^2 + KEh + E)w_0(f, h) \\ &\leqq Mw_0(f, h) \end{aligned}$$

where $M$ is some constant independent of $h$ and thus the proof is complete.

N.B.I have solved the same problem but for the general case when $f \in C^r$, where $r$ is finite positive integer and this will appear in the next paper under the same title.

The problem of the non linear ordinary differential equation of the $n$-th order has been solved by the same method and also when $f \in C^r$ and this will appear in the future.

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Kuwait Institute of Applied Technology
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