SOME EXAMPLES FOR A NEW ERROR ESTIMATES OF GAUSS-JACOBI QUADRATURE FORMULAE BASED ON THE CHEBYSHEV ROOTS

By

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1. *Introduction.* Let $d\alpha$ be a nonnegative measure on the whole or a part of the real line as its support. We assume that the support of $d\alpha$ contains infinitely many points. Then there exists a uniquely determined sequence of orthonormal polynomials $\{p_n(d\alpha; x)\}$ with respect to this weight, they are determined by the properties that

(a) $p_n(d\alpha; x) = \gamma_n(d\alpha)x^n + \dots$ is a polynomial of degree n and $\gamma_n(d\alpha) > 0$.

(b) we have

$$\int p_n(d\alpha)\, p_m(d\alpha)\, d\alpha = \delta_{mn}$$

where δ_{mn} is the Kronecker symbol.

It is wellknown that all zeroes $x_{kn} = x_{kn}(d\alpha)$ are real and are contained in the smallest interval overlapping the support of $d\alpha$.

The interpolatory quadrature formula

$$Q_n(d\alpha; f) \overset{\text{def}}{=} \sum_{k=1}^{n} \lambda_n(d\alpha; x_{kn}) f(x_{kn}) \quad (\sim \int f d\alpha) \tag{1.1}$$

has the property that $Q_n(d\alpha; p_{2n-1}) = \int p_{2n-1}\, d\alpha$ for every polynomial of degree $2n-1$ at most.

The Cotes numbers $\lambda_n(d\alpha; x_{kn})$ of this formula are called Christoffel numbers and are represented by

$$\lambda_n^{-1}(d\alpha; x) = \sum_{\nu=0}^{n-1} p_\nu^2(d\alpha; x)\,. \tag{1.2}$$

Usually (1.1) is called (after their first inventors) the Gauss-Jacobi quadrature formula. The nodes $x_{kn} = x_{kn}(d\alpha)$ are called the Gaussian abscissas with respect to $d\alpha$.

2. *The "classical" error estimate.* It is wellknown A. A. Markov's following classical result: If $f^{(2n)}$ is continuous, then

$$\int f d\alpha - Q_n(d\alpha; f) = \frac{f^{(2n)}(\xi)}{(2_n)!\, \gamma_n^2(d\alpha)} \quad (\xi \in \text{support of } d\alpha) \tag{2.1}$$

([2], (2.7.9)).

As the analytic treatment of the error estimate we mention McNamee's method for the measure $d\alpha(x) = dx$.

Let B is a simply connected region in the complex z plane. Suppose that $f(z)$ is analytic in B. Denoting the n^{th} Legendre polynomial by $P_n(z)$ we obtain

$$\int_{-1}^{1} f(x)\, dx - Q_n(dx; f) = \frac{1}{i\pi} \int_C \frac{f(t)\, Q_n(t)}{P_n(t)} dt \quad (-1 \leqq t \leqq 1) \tag{2.2}$$

where

$$Q_n(t) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(z)}{t - z} dz$$

are commonly called the Legendre functions of the second kind, C is a simple contour contained in B and containing the roots $z_1, z_2, \ldots, z_n$ of the Legendre polynomial $P_n(z)$. Using an asymptotic expression for $Q_n(t)/P_n(t)$ and taking a very large contour C we get an upper bound for the integral on the right of (2.2) ([2], 4. 6).

3. *A new error estimate.* In [1] the first named author proved, among others, the following result.

In what follows let the support of $d\alpha$ be [1, 1]. We assume further that

$$\log \alpha'(\cos \vartheta) \in \mathcal{L}[-\pi, \pi]. \tag{3.1}$$

For any such $d\alpha$ we set $g(\vartheta) = \alpha'(\cos \vartheta)\, |\sin \vartheta|$ and

$$D(d\alpha; w) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log g(\vartheta) \frac{1 + we^{-i\vartheta}}{1 - we^{-i\vartheta}} d\vartheta \right\}. \tag{3.2}$$

$D(d\alpha; w)$ is analytic in the unit circle and

$$D(d\alpha; 0) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log [\alpha'(\cos \vartheta)\, |\sin \vartheta|]\, d\vartheta \right\}. \tag{3.3}$$

With these notations we have

THEOREM 3.1 (G. FREUD) If $f(z)$ is analytic in $|z + \sqrt{z^2 - 1}| \leqq r$ and $d\alpha$ satisfies (3.1) then

$$\left| \int_{-1}^{1} f d\alpha - Q_n(d\alpha; f) \right| \leqq$$

$$\leqq \frac{2M_e(f;r)}{r^{2n+1}+r^{2n-1}} \int_{e(r)} |D(d\alpha; z-\sqrt{z^2-1}|^2 \, |dz| \, [1+o(1)] \, . \tag{3.4}$$

Here

$$M_e(f;r) = \max_{|z+\sqrt{z^2-1}|\leqq r} |f(z)| \tag{3.5}$$

and $e(r)$ denotes the ellipse $|z+\sqrt{z^2-1}| = r$ ([1], (15)).

The aim of this paper is to give some numerical examples for the above mentioned theorem when the orthogonal polynomials are the Chebyshev polynomials, i.e.

$$d\alpha = (1-x^2)^{-1/2} dx, \; x_{2n} = \cos \frac{2k-1}{2n} \pi \, , \tag{3.6}$$

$$\lambda_n(x_{kn}) = \frac{\pi}{n}, \; D(w) \equiv 1$$

([3], 12.1 and 15.3).

Then from the asymptotic formula (3.4) we obtain the following

THEOREM 3.2 If $f(z)$ is analytic in $|z+\sqrt{z^2-1}| \leqq r$ then

$$\left| \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx - \frac{\pi}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k-1}{2n} \pi\right) \right| \leqq \frac{2M_e(f;r)}{r^{2n}+r^{2n-2}} \pi \, . \tag{3.7}$$

(Indeed, now the $o(1)$ vanishes, further $\int_{e(r)} |D|^2 \, |dz| = r\pi$; see the proof from [1].)

4. *Numerical examples*. In this part we consider some functions $f(x)$ for which (3.7) gives essentially better error estimations than (2.1).

First of all, we have from (2.1) and (3.6)

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx - \frac{\pi}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k-1}{2n} \pi\right) = \tag{3.8}$$

$$= \frac{\pi}{(2n)! \, 2^{2n-1}} f^{(2n)}(\xi) \quad (-1<\xi<1)$$

([4], V., 4. §).

Let

$$(3.9)\qquad f_s(z) = \frac{1}{[z-(1+2\varepsilon)]^s} \qquad (s>0,\ \text{integer}\ \varepsilon>0).$$

These functions are analytic in the ellipse $e(r_1)$ where $r_1 = 1+\varepsilon+\sqrt{2\varepsilon+\varepsilon^2}$.* By (3.9) we have

$$(3.10)\qquad M_e(f_s; r_1) = |f_s(1+\varepsilon)| = \frac{1}{\varepsilon^s} \qquad (s = 1, 2, 3, \dots),$$

further considering that

$$(3.11)\qquad f_s^{(k)}(z) = (-1)^k \frac{(s+k-1)!}{(s-1)!\,[z-(1+\varepsilon)]^{s+k}} \qquad (k = 0, 1, 2, \dots)$$

we get the relation

$$(3.12)\qquad |f_s^{(2n)}(\xi)| < |f_s^{(2n)}(1)| = \frac{(s+2n-1)!}{(s-1)!\,(2\varepsilon)^{s+2n}}$$

$$(s = 1, 2, 3, \dots;\ n = 1, 2, 3, \dots)$$

for any $1<\xi<1$.
So using (3.7) and (3.10) we get

$$(3.13)\qquad R_1 \overset{\text{def}}{=} \frac{2M_e(f_s; r_1)}{r_1^{2n}+r_1^{2n-2}}\pi = \frac{2\pi}{\varepsilon^s (r_1^2+1)\, r_1^{2n-2}} \qquad (s = 1, 2, 3, \dots)$$

further, by (3.8) and (3.12)

$$(3.14)\qquad R_2 \overset{\text{def}}{=} \frac{\pi}{(2n)!\,2^{2n-1}} f_s^{(2n)}(\xi) <$$

$$< \begin{cases} \dfrac{\pi}{2^{2n-1}} \cdot \dfrac{1}{(2\varepsilon)^{2n+1}} & (s = 1) \\[2ex] \dfrac{\pi}{2^{2n-1}} \cdot \dfrac{(2n+1)(2n+2)\dots(2n+s-1)}{(s-1)!\,(2\varepsilon)^{2n+s}} & (s = 2, 3, 4, \dots). \end{cases}$$

Now we have the tools to compare the formulae (3.7) and (3.8) for the functions $f_s(z)$.

* We have r_1 from $1+\varepsilon = \dfrac{1}{2}\left(r_1+\dfrac{1}{r_1}\right)$. The focii of our ellipse are -1 and 1, its axes are of length $\dfrac{1}{2}\left(r_1+\dfrac{1}{r_1}\right)$ and $\dfrac{1}{2}\left(r_1-\dfrac{1}{r_1}\right)$.

By these tools we made a FORTRAN program for the computer ODRA 1304 of the Eötvös Loránd University, Budapest. This program computes the GAUSS SUM $\frac{\pi}{n}\sum_{k=1}^{n} f_s\left(\cos\frac{2k-1}{2n}\pi\right)$ for $n = 2, 3, 6, 9, 18, 27$; upper bounds for R_1 (NEW ERROR-EST) and R_2 (OLD ERROR-EST) their differences $R_2 - R_1$ (DIFFERENCE). Our program works with various s and ε. (In the program S and EPS)

Here we mention that we can compute analogous results for other functions $f(z)$ changing the segments THEFUNCT and ERROR AND PRINT and the 8 FORMAT in MASTER.

Finally we publish our program and the results.

We computed $f_s(x)$ for $s = 1, 2, 4, 6$ and 7, with $\varepsilon = 0.3, 0.4, 0.7, 1.0$ and 2.5 (If $s = 1$ or $s = 2$ then also for $\varepsilon = 0.1$ and 0.2). We can see that for small ε ($\varepsilon < 1$) $R_1 < R_2$ but in the case $\varepsilon = 2.5$, $R_2 < R_1$. If $\varepsilon = 1$ then $R_2 < R_1$ for "small" s and $R_1 < R_2$ for "large" s. We have to notice that R_1 is a very good and usable error estimation even if $R_2 < R_1$ (if n is large enough), but this remark is not true for R_2. (See, e.g. $\varepsilon = 1, 2.5$; $n = 18, 27$; s is arbitrary; or $\varepsilon = 0.1, 0.2$; $n = 2, 3, 6, 9, 18, 27$.)

REFERENCES

[1] *G. Freud,* Error estimates for Gauss-Jacobi quadrature formulae in: Topics in Numerical Analysis, Ed. John J. H. Miller (Academic Press, New York and London, 1973), 113–121.

[2] *P. J. Davies, D. Rabinowitz,* Numerical Integration, Blaisdell (Walthem, Massachusetts) 1967.

[3] *G. Szegő,* Orthogonal polynomials, Amer. Math. Soc. 1959.

[4] *I. P. Natanson,* Constructive Function Theory, New York. Ungar, 1964.

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RUN BY GEORGE 2/MK9B ON 08/02/73 AT 16.50

```
            JOB GJCN,1MNGO77,TESI
            FORTRUM GJCN
            FORTCOMP GJCN, ,
            ****
DOCUMENT GJCN     08/02/73 AT 16.50
```

FORTRAN COMPILATION BY #XFAM MK 4E DATE 08/02/73 TIME 16/51/11

```
    LIST(LP)
    PROGRAM(QUAD)
    INPUT 1=CRO
    OUTPUT 2=LPO
    END

    MASTER CHIEF FOR GAUSS
    DIMENSION F(18),G(27)
    COMMON /SET/FN,IS,EPS,PI,GAUSS,IWAY
    READ(1,13)ISEND

 13 FORMAT(I0)
    FN=2
    PI=3.1415926536
    WRITE(2,8)

  8 FORMAT(1H1/////,37X,46HNEW ERROR ESTIMATE FOR GAUSS-JACOBI
    QUADRAT XURE/////,38X,45HF(X)=1/(X-(1+2*EPS))**S (EPS AND S ARE
    FIXED)//)

 17 READ(1,13)IS,IEPS
    IE=1

 15 READ(1,9)EPS

  9 FORMAT(FO.O)
    SF,SG=0
    DO 10 I=1,18
    F(I)=THEFUNCT(COS((2.0*I-1.0)/36.0*PI))

 10 SF=SF+F(I)
    DO 11 I=1,27
    G(I)=THEFUNCT(COS((2.0*I-1.0)/54.0*PI))

 11 SG=SG+G(I)
    IWAY=1

  1 FS,S=F(5)+F(14)
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```
12 GAUSS=S*PI/FN
   CALL ERROR AND PRINT
   IWAY=IWAY+1
   GO TO (1,2,3,4,5,6,7)IWAY

 2 FN=3
   GS,S=G(5)+G(14)+G(23)
   GO TO 12

 3 FN=6
   FS,S=FS+F(2)+F(8)+F(11)+F(17)
   GO TO 12

 4 FN=9
   GS,S=GS+G(2)+G(8)+G(11)+G(17)+G(20)+G(26)
   GO TO 12

 5 FN=18
   S=SF
   GO TO 12

 6 FN=27
   S=SG
   GO TO 12

 7 FN=2
   IF(IE-IEPS)0,14,14
   IE=IE+1
   GO TO 15

14 IF(IS-ISEND)0,16,16
   GO TO 17

16 STOP

   END

END OF SEGMENT, LENGTH   302, NAME CHIEFFORGAUSS

   FUNCTION THEFUNCT(X)

   COMMON/SET/FN,IS,EPS

   THEFUNCT=1.0/(X-1.0-2.0*EPS)**IS

   RETURN

   END
```

```
END OF SEGMENT, LENGTH 23, NAME THEFUNCT

   SUBROUTINE ERROR AND PRONT
   COMMON/SET/FN,IS,EPS,PI,GAUSS,IWAY
   N=INT(FN+0.1)
   R=1+EPS+SQRT(2*EPS+EPS*EPS)
   ESTNEW=2*PI/EPS**IS/R**(2*N-2)/(1.+R**2)
   H1,H2=1
   IF(IS-1)0,1,0
   DO 2 J=1,IS-1
   H1=H1*(2*N+J)
 2 H2=H2*J
   H1=H1/H2
 1 ESTOLD=H1*PI/2.**(2*N-1)/(2*EPS)**(2*N+IS)
   D=ESTOLD-ESTNEW
   IF(IWAY-1)3,0,3
   WRITE(2,4)
 4 FORMAT(1HO,//,4X,2HS=,10X,4HEPS=,)
   WRITE(2,5)IS,EPS
 5 FORMAT(1H+,6X,I4,10X,E9.2/)
   WRITE(2,6)
 6 FORMAT(11X,12HNODES-NUMBER,10X,9HGAUSS-SUM,10X,14HNEW
   ERROR-EST., X10X, 14HOLD ERROR-EST., 10X,10HDIFFERENCE)
 3 WRITE(2,7)N,GAUSS,ESTNEW,ESTOLD,D
 7 FORDAT(16X,12,11X,E17.10,8X,E9.2,15X,E9.2,13,E9.2)
   RETURN
   END

END OF SEGMENT, LENGTH   167, NAME ERRORANDPRINT
    FINISH
PROGRAM NAME QUAD, CORE  3756,LOWER AREA  469,PROGRAM  2818
END OF COMPILATION - NO ERRORS
DOCUMENT GJCN       [08/02/73 AT 16.55
```

NEW ERROR ESTIMATE FOR GAUSS – JACOBI QUADRATURE

F(X) = 1/(X – (1 + 2 * EPS)) * * S (EPS AND S ARE FIXED)

S = 1 EPS = 0.10E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	–0.4010543813E 01	0.75E 04	0.12E 04	0.12E 04
3	–0.4515090891E 01	0.31E 01	0.77E 04	0.77E 04
6	–0.4730724788E 01	0.22E 00	0.19E 07	0.19E 07
9	–0.4735999929E 01	0.15E – 01	0.46E 09	0.46E 09
18	–0.4736129124E 01	0.52E – 05	0.67E 16	0.67E 16
27	–0.4736129126E 01	0.18E 08	0.97E 23	0.97E 23

S = 1 EPS = 0.20E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	–0.3012486106E 01	0.20E 01	0.38E 02	0.36E 02
3	–0.3171265312E 01	0.58E 00	0.60E 02	0.59E 02
6	–0.3206180239E 01	0.14E – 01	0.23E 03	0.23E 03
9	–0.3206373506E 01	0.33E – 03	0.87E 03	0.87E 03
18	–0.3206374575E 01	0.45E – 08	0.48E 05	0.48E 05
27	–0.3206374575E 01	0.62E – 13	0.27E 07	0.27E 07

S = 1 EPS = 0.30E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	–0.2440071964E 01	0.83E 00	0.51E 01	0.42E 01
3	–0.2505897455E 01	0.18E 00	0.35E 01	0.33E 01
6	–0.2515269567E 01	0.20E – 02	0.12E 01	0.12E 01
9	–0.2515287125E 01	0.21E – 04	0.39E 00	0.39E 00
18	–0.2515287158E 01	0.26E – 10	0.15E – 01	0.15E – 01
27	–0.2515287158E 01	0.31E – 16	0.55E – 03	0.55E – 03

S = 1 EPS = 0.40E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	– 0.2063819991E 01	0.42E 00	0.12E 01	0.78E 00
3	– 0.2095796973E 01	0.73E – 01	0.47E 00	0.39E 00
6	– 0.2099062338E 01	0.40E – 03	0.28E – 01	0.27E – 01
9	– 0.2099064883E 01	0.22E – 05	0.17E – 02	0.17E – 02
18	– 0.2099064886E 01	0.37E – 12	0.35E – 06	0.35E – 06
27	– 0.2099064885E 01	0.62E – 19	0.75E – 10	0.75E – 10

S = 1 EPS = 0.70E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	– 0.1433426306E 01	0.91E – 01	0.73E – 01	– 0.18E – 01
3	– 0.1439635356E 01	0.96E – 02	0.93E – 02	– 0.29E – 03
6	– 0.1439946598E 01	0.11E – 04	0.19E – 04	0.80E – 05
9	– 0.1439946632E 01	0.13E – 07	0.40E – 07	0.27E – 07
18	– 0.1439946632E 01	0.22E – 16	0.36E – 15	0.34E – 15
27	– 0.1439946632E 01	0.37E – 25	0.32E – 23	0.32E – 23

S = 1 EPS = 0.10E 01

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	– 0.1108797407E 01	0.30E – 01	0.12E – 01	– 0.18E – 01
3	– 0.1110664069E 01	0.22E – 02	0.77E – 03	– 0.14E – 02
6	– 0.1110720733E 01	0.80E – 06	0.19E – 06	– 0.62E – 06
9	– 0.1110720735E 01	0.30E – 09	0.46E – 10	– 0.25E – 09
18	– 0.1110720735E 01	0.15E – 19	0.67E – 21	– 0.14E – 19
27	– 0.1110720735E 01	0.76E – 30	0.97E – 32	– 0.75E – 30

S = 1 EPS = 0.25E 01

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	– 0.5309734062E 00	0.11E – 02	0.13E – 03	– 0.99E – 03
3	– 0.5310257086E 00	0.24E – 04	0.13E – 05	– 0.22E – 04
6	– 0.5310260796E 00	0.23E – 09	0.13E – 11	– 0.23E – 09
9	– 0.5310260796E 00	0.22E – 14	0.13E – 17	– 0.22E – 14
18	– 0.5310260796E 00	0.20E – 29	0.13E – 35	– 0.20E – 29
27	– 0.5310260796E 00	0.18E – 44	0.13E – 53	– 0.18E – 44

S = 2 EPS = 0.10E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.6897566487E 01	0.75E 02	0.31E 05	0.31E 05
3	0.1036117405E 02	0.31E 02	0.27E 06	0.27E 06
6	0.1280426433E 02	0.22E 01	0.12E 09	0.12E 09
9	0.1291285759E 02	0.15E 00	0.43E 11	0.43E 11
18	0.1291671570E 02	0.52E – 04	0.12E 19	0.12E 19
27	0.1291671580E 02	0.18E – 07	0.27E 26	0.27E 26

S = 2 EPS = 0.20E 00

NODES – NUMBER	GASSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.3625594825E 01	0.10E 02	0.48E 03	0.47E 03
3	0.4410939562E 01	0.29E 01	0.10E 04	0.10E 04
6	0.4673299460E 01	0.70E – 01	0.74E 04	0.74E 04
9	0.4675941708E 01	0.17E – 02	0.41E 05	0.41E 05
18	0.4675962923E 01	0.23E – 07	0.45E 07	0.45E 07
27	0.4675962922E 01	0.31E – 12	0.37E 09	0.37E 09

S = 2 EPS = 0.30E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.2265358073E 01	0.28E 01	0.42E 02	0.39E 02
3	0.2525128754E 01	0.61E 00	0.41E 02	0.40E 02
6	0.2579594643E 01	0.65E – 02	0.25E 02	0.25E 02
9	0.2579781193E 01	0.70E – 04	0.12E 02	0.12E 02
18	0.2579781701E 01	0.85E – 10	0.91E 00	0.91E 00
27	0.2579781701E 01	0.10E – 15	0.51E – 01	0.51E – 01

S = 2 EPS = 0.40E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.1565021648E 01	0.10E 01	0.75E 01	0.64E 01
3	0.1671031972E 01	0.18E 00	0.41E 01	0.39E 01
6	0.1686726094E 01	0.10E – 02	0.45E 00	0.45E 00
9	0.1686748543E 01	0.56E – 05	0.39E – 01	0.39E – 01
18	0.1686748569E 01	0.93E – 12	0.16E – 04	0.16E – 04
27	0.1686748569E 01	0.15E – 18	0.51E – 08	0.51E – 08

S= 2 EPS= 0.70E 00

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.7108086719E 00	0.13E 00	0.26E 00	0.13E 00
3	0.7250106203E 00	0.14E–01	0.47E–01	0.33E–01
6	0.7260233098E 00	0.16E–04	0.18E–03	0.16E–03
9	0.7260235159E 00	0.19E–07	0.54E–06	0.53E–06
18	0.7260235119E 00	0.32E–16	0.95E–14	0.94E–14
27	0.7260235119E 00	0.53E–25	0.13E–21	0.13E–21

S= 2 EPS= 0.10E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.4130813870E 00	0.30E–01	0.31E–01	0.46E–03
3	0.4163788243E 00	0.22E–02	0.27E–02	0.51E–03
6	0.4165202688E 00	0.80E–06	0.12E–05	0.41E–06
9	0.4165202755E 00	0.30E–09	0.43E–09	0.14E–09
18	0.4165202755E 00	0.15E–19	0.12E–19	–0.28E–20
27	0.4165202755E 00	0.76E–30	0.27E–30	–0.50E–30

S= 2 EPS= 0.25E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.9098840060E–01	0.45E–03	0.13E–03	–0.32E–03
3	0.9103260238E–01	0.95E–05	0.18E–05	–0.77E–05
6	0.9103304221E–01	0.92E–10	0.33E–11	–0.88E–10
9	0.9103304221E–01	0.88E–15	0.48E–17	–0.88E–15
18	0.9103304221E–01	0.79E–30	0.93E–35	–0.79E–30
27	0.9103304221E–01	0.71E–45	0.14E–52	–0.71E–45

S= 4 EPS= 0.30E 00

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.2526722301E 01	0.31E 02	0.82E 03	0.79E 03
3	0.3796421586E 01	0.68E 01	0.14E 04	0.14E 04
6	0.4299208443E 01	0.73E–01	0.25E 04	0.25E 04
9	0.4303851921E 01	0.78E–03	0.24E 04	0.24E 04
18	0.4303876440E 01	0.95E–09	0.63E 03	0.63E 03
27	0.4303876440E 01	0.12E–14	0.75E 02	0.75E 02

S= 4 EPS= 0.40E 00

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.1140813212E 01	0.65E 01	0.82E 02	0.75E 02
3	0.1496706152E 01	0.11E 01	0.77E 02	0.76E 02
6	0.1593049463E 01	0.63E–02	0.25E 02	0.25E 02
9	0.1593427935E 01	0.35E–04	0.43E 01	0.43E 01
18	0.1593428774E 01	0.58E–11	0.63E–02	0.63E–02
27	0.1593428774E 01	0.97E–18	0.43E–05	0.43E–05

S= 4 EPS= 0.70E 00

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.2081037719E 00	0.26E 00	0.93E 00	0.67E 00
3	0.2298948864E 00	0.28E–01	0.29E 00	0.26E 00
6	0.2326325701E 00	0.33E–04	0.32E–02	0.32E–02
9	0.2326441133E 00	0.39E–07	0.19E–04	0.19E–04
18	0.2326341138E 00	0.65E–16	0.12E–11	0.12E–11
27	0.2326341138E 00	0.11E–24	0.34E–19	0.34E–19

S= 4 EPS= 0.10E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERRPOR–EST	DIFFERENCE
2	0.6514814808E–01	0.30E–01	0.54E–01	0.23E–01
3	0.6811387732E–01	0.22E–02	0.81E–02	0.59E–02
6	0.6833532780E–01	0.80E–06	0.11E–04	0.98E–05
9	0.6833535769E–01	0.30E–09	0.76E–08	0.73E–08
18	0.6833535769E–01	0.15E–19	0.76E–18	0.74E–18
27	0.6833535769E–01	0.76E–30	0.35E–28	0.35E–28

S= 4 EPS= 0.25E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	0.2777671565E–02	0.71E–04	0.35E–04	–0.36E–04
3	0.2786573449E–02	0.15E–05	0.84E–06	–0.67E–06
6	0.2786725782E–02	0.15E–10	0.46E–11	–0.10E–10
9	0.2786725782E–02	0.14E–15	0.13E–16	–0.13E–15
18	0.2786725782E–02	0.13E–30	0.92E–34	–0.13E–30
27	0.2786725782E–02	0.11E–45	0.29E–51	–0.11E–45

S = 6 EPS = 0.30E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.3110138186E 01	0.34E 03	0.82E 04	0.78E 04
3	0.6765032564E 01	0.75E 02	0.21E 05	0.21E 05
6	0.9197925151E 01	0.81E 00	0.93E 05	0.93E 05
9	0.9246654393E 01	0.86E – 02	0.17E 06	0.17E 06
18	0.9247107753E 01	0.11E – 07	0.14E 06	0.14E 06
27	0.9247107754E 01	0.13E – 13	0.36E 05	0.36E 05

S = 6 EPS = 0.40E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.9281609543E 00	0.41E 02	0.46E 03	0.42E 03
3	0.1611382744E 01	0.72E 01	0.66E 03	0.65E 03
6	0.1911648528E 01	0.40E – 01	0.53E 03	0.53E 03
9	0.1914280935E 01	0.22E – 03	0.17E 03	0.17E 03
18	0.1914291356E 01	0.36E – 10	0.81E 00	0.81E 00
27	0.1914291356E 01	0.61E – 17	0.11E – 02	0.11E – 02

S = 6 EPS = 0.70E 00

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.6847903512E – 01	0.54E 00	0.17E 01	0.12E 01
3	0.8671730358E – 01	0.57E – 01	0.80E 00	0.74E 00
6	0.9029812695E – 01	0.68E – 04	0.22E – 01	0.22E – 01
9	0.9030284801E – 01	0.80E – 07	0.25E – 03	0.25E – 03
18	0.9030285068E – 01	0.13E – 15	0.50E – 10	0.50E – 10
27	0.9030285069E – 01	0.22E – 24	0.30E – 17	0.30E – 17

S = 6 EPS = 0.10E 01

NODES – NUMBER	GAUSS – SUM	NEW ERROR – EST	OLD ERROR – EST	DIFFERENCE
2	0.1141499552E – 01	0.30E – 01	0.48E – 01	0.18E – 01
3	0.1283915095E – 01	0.22E – 02	0.11E – 01	0.89E – 02
6	0.1300349444E – 01	0.80E – 06	0.36E – 04	0.35E – 04
9	0.1300354741E – 01	0.30E – 09	0.48E – 07	0.48E – 07
18	0.1300354742E – 01	0.15E – 19	0.16E – 16	0.16E – 16
27	0.1300354742E – 01	0.76E – 30	0.15E – 26	0.15E – 26

S = 6 EPS = 0.25E 01

NODES−NUMBER	GAUSS−SUM	NEW ERROR−EST	OLD ERROR−EST	DIFFERENCE
2	0.8869797554E−04	0.11E−04	0.51E−05	−0.64E−05
3	0.8962857248E−04	0.24E−06	0.19E−06	−0.57E−07
6	0.8965294933E−04	0.23E−11	0.25E−11	0.14E−12
9	0.8965294945E−04	0.23E−16	0.14E−16	−0.91E−17
18	0.8935294944E−04	0.20E−31	0.30E−33	−0.20E−31
27	0.8935294944E−04	0.18E−46	0.20E−50	−0.18E−46

S = 7 EPS = 0.30E 00

NODES−NUMBER	GAUSS−SUM	NEW ERROR−EST	OLD ERROR−EST	DIFFERENCE
2	−0.3476063116E 01	0.11E 04	0.23E 05	0.22E 05
3	−0.9166499101E 01	0.25E 03	0.69E 05	0.69E 05
6	−0.1393958314E 02	0.27E 01	0.47E 06	0.47E 06
9	−0.1407183933E 02	0.29E−01	0.11E 07	0.11E 07
18	−0.1407342309E 02	0.35E−07	0.17E 07	0.17E 07
27	−0.1407342309E 02	0 43E−13	0.60E 06	0.60E 06

S = 7 EPS = 0.40E 00

NODES−NUMBER	GAUSS−SUM	NEW ERROR−EST	OLD ERROR−EST	DIFFERENCE
2	−0.8460048399E 00	0.10E 03	0.96E 03	0.86E 03
3	−0.1707406989E 01	0.18E 02	0.17E 04	0.16E 04
6	−0.2175631557E 01	0.99E−01	0.20E 04	0.20E 04
9	−0.21814040693 01	0.54E−03	0.85E 03	0.85E 03
18	−0.2181433734E 01	0.91E−10	0.70E 01	0.70E 01
27	−0.2181433734E 01	0.15E−16	0.14E−01	0.14E−01

S = 7 EPS = 0.70E 00

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	–0.3998152220E–01	0.77E 00	0.20E 01	0.13E 01
3	–0.5494380604E–01	0.82E–01	0.11E 01	0.11E 01
6	–0.5850480852E–01	0.97E–04	0.48E–01	0.48E–01
9	–0.5851160154E–01	0.11E–06	0.72E–03	0.72E–03
18	–0.5851160660E–01	0.19E–15	0.25E–09	0.25E–09
27	–0.5851160660E–01	0.31E–24	0.21E–16	0.21E–16

S = 7 EPS = 0.10E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	–0.4877729977E–02	0.30E–01	0.40E–01	0.10E–01
3	–0.5756373140E–02	0.22E–02	0.11E–01	0.89E–02
6	–0.5878602806E–02	0.80E–06	0.54E–04	0.54E–04
9	–0.5878660570E–02	0.30E–09	0.96E–07	0.96E–07
18	–0.5878660580E–02	0.15E–19	0.55E–16	0.55E–16
27	–0.5878660581E–02	0.76E–30	0.76E–26	0.76E–26

S = 7 EPS = 0.25E 01

NODES–NUMBER	GAUSS–SUM	NEW ERROR–EST	OLD ERROR–EST	DIFFERENCE
2	–0.1607056052E–04	0.46E–05	0.17E–05	–0.29E–05
3	–0.1633577395E–04	0.97E–07	0.74E–07	–0.23E–07
6	–0.1634411506E–04	0.94E–12	0.15E–11	0.56E–12
9	–0.1634411512E–04	0.90E–17	0.11E–16	0.18E–17
18	–0.1634411512E–04	0.81E–32	0.42E–33	–0.77E–32
27	–0.1634411512E–04	0.73E–47	0.40E–50	–0.73E–47