A LINEAR OLIGOPOLY MODEL

By

FERENC SZIDAROVSZKY

(Received November 8, 1975)

This paper presents a game-theoretical model of a multiproduct economy, where the production costs of producer groups and the price functions are linear. First an existence theorem is given for the Nash-equilibrium point of this game, and in a further special case also the uniqueness of the equilibrium point is proven. In this case a computational procedure for finding the equilibrium point is proposed.

1. Introduction

Let N denote the number of groups of producers, and let the number of the members of the group k $(1 \leqq k \leqq N)$ be denoted by i_k. Assume that the number of different products is M.

Let $x_{ki}^{(m)}$ denote the production level of the product m $(1 \leqq m \leqq M)$ manufactured by the i^{th} player of group k and assume that the production levels $x_{ki}^{(m)}$ are bounded by capacity limits $L_{ki}^{(m)}$. It is also assumed that the price functions of the products depend on the total amounts of the different products. Thus the set of strategies of group k is given by

$$S_k = \mathop{X}_{i=1}^{i_k} \mathop{X}_{m=1}^{M} [0, L_{ki}^{(m)}], \tag{1.1}$$

and the pay-off function of this group can be expressed as

$$\varphi_k(x_1, \dots, x_N) = \sum_{m=1}^{M} s_k^{(m)} f_m(s^{(1)} \dots, s^{(M)}) - K_k(\mathbf{x}_k), \tag{1.2}$$

where f_m is the price of product m, K_k is the cost of group k,

$$\mathbf{x}_l = (x_{l1}^{(1)} \dots, x_{l1}^{(M)} \dots, x_{li_l}^{(1)} \dots, x_{ki_l}^{(M)}) \quad (1 \leq l \leq N),$$

$$s_k^{(m)} = \sum_{i=1}^{i_k} x_{ki}^{(m)} \quad \text{and} \quad s^{(m)} = \sum_{l=1}^{N} s_l^{(m)} \quad (1 \leq m \leq M).$$

Definition I. A Nash equilibrium point of the game defined by the sets of strategies (1.1) and pay-off functions (1.2) is a vector $x^* = (x_1^*, \ldots, x_N^*)$ such that for $k = 1, 2, \ldots, N$ $\mathbf{x}_k^* \in S_k$ and

$$\varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*) \geqq \varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N^*) \quad \mathbf{x}_k \in S_k. \tag{1.3}$$

In the special case of $M = 1$, $i_k = 1$ $(k = 1, \ldots, N)$ an existence and uniqueness theorem can be found in Burger's book [2], where the price function is strictly monoton decreasing and concave, the cost functions are strictly increasing and convex. It is also assumed that the cost functions are identical to each other, and both the price function and the cost function are twice differentiable. With dropping the symmetricity assumption of the game Szidarovszky proved the existence and uniqueness of the equilibrium point and in his paper [7] an iteration algorithm is introduced for computing the equilibrium point. The existence of the equilibrium point in the special case of concave pay-off functions can be proven by using the Nikaido—Isoda theorem, but the proof is based on fixed point theorems and does not provide on effective procedure for finding the equilibrium point. Also in the concave case the existence of the equilibrium point was proven by Frank and Quandt [3] and the uniqueness was shown by Opitz [5]. The group equilibrium problem in the strictly concave case was solved in the paper [8]. The results of paper [6] give the connections of the noncooperative N-person games with mathematical programming problems.

2. The existence theorem

Let us introduce the notation

$$L_k^{(m)} = \sum_{i=1}^{i_k} L_k^{(m)}, \quad L^{(m)} = \sum_{k=1}^{N} L_k^{(m)}, \quad (1 \le k \le N, \ 1 \le m \le M)$$

and let us make the following assumptions:

A) For $\mu = 1, 2, \ldots M$ and $(s^{(1)}, \ldots, s^{(M)}) \notin D$, $f_\mu(s^{(1)}, \ldots, s^{(M)}) = 0$, where $D \subset R^M$ is a convex, closed set.

B) For $\mu = 1, 2, \ldots, M$ and $(s^{(1)}, \ldots, s^{(M)}) \in D$,

$$f_\mu(s^{(1)}, \ldots, s^{(M)}) = \sum_{m=1}^{M} a_\mu^{(m)} s^{(m)} + b_\mu. \tag{2.1}$$

C) For abitrary $(s^{(1)}, \ldots, s^{(M)}) \in D$ and

$$0 \le \tilde{s}^{(m)} \le s^{(m)} \quad (1 \le m \le M), \quad (s^{(1)}, \ldots, s^{(m-1)}, \ \tilde{s}^{(m)}, s^{(m+1)}, \ldots, s^{(M)}) \in D.$$

D) The cost functions have the form

$$K_k(\mathbf{x}_k) = \sum_{i=1}^{i_k} \sum_{m=1}^{M} A_{ki}^{(m)} x_{ki}^{(m)} + B_k \quad (k = 1, 2, \ldots, N), \tag{2.2}$$

where the coefficients $A_{ki}^{(m)}$ are positive numbers.

It is easy to verify that the above assumptions are not sufficient for the concavity of the pay-off functions.

First we prove the following theorem.

Theorem 1. Let $\mathbf{A} = (a_{\mu}^{(m)})_{\mu, m=1}^{M}$, and assume that $\mathbf{A}+\mathbf{A}^T$ is a negative semidefinite matrix. Than the game has at least one equilibrium point.

Proof. The proof consists of several stages.

a) First we prove that φ_k is concave in $\mathbf{x}_k$.
Simple calculation shows that the Hessian of φ_k is equal to $\mathbf{A}+\mathbf{A}^T$, which implies the assertion.

b) Let $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ be an equilibrium point, where

$$\mathbf{x}_k^* = (x_{k1}^{(1)*}, \ldots, x_{k1}^{(M)*}, \ldots, x_{ki_k}^{(1)*}, \ldots, x_{ki_k}^{(M)*}) \quad (1 \le k \le N).$$

We prove that

$$\mathbf{s}^* = \left(\sum_{k=1}^{N} \sum_{i=1}^{i_k} x_{ki}^{(1)*}, \ldots, \sum_{k=1}^{N} \sum_{i=1}^{i_k} x_{ki}^{(M)*} \right) \in D.$$

Assume that $s^* \notin D$, and let

$$S = \Big\{ (x_{11}^{(1)}, \ldots, x_{11}^{(M)}, \ldots, x_{Ni_N}^{(1)}, \ldots x_{Ni_N}^{(M)}) \mid \; 0 \le x_{ki}^{(m)} \le L_{ki}^{(m)} \;\; \forall\, k, i, m;$$

$$\left(\sum_{k=1}^{N} \sum_{i=1}^{i_k} x_{ki}^{(1)}, \ldots, \sum_{k=1}^{N} \sum_{i=1}^{i_k} x_{ki}^{(M)} \right) \in D \Big\}. \tag{2.3}$$

Obviously S is convex. For $S = \emptyset$ it is easy to verify that $\varphi_k(\mathbf{x}) = -K_k(\mathbf{x}_k)$ $\left(\mathbf{x} \in \mathop{X}_{k=1}^{N} S_k\right)$, consequently $\mathbf{x} = \mathbf{0}$ is the only equilibrium point. Let us now assume that $S \ne \emptyset$. Then *c)* implies that $\mathbf{0} \in S$, and therefore at least one component $x_{ki}^{(m)*}$ of $\mathbf{s}^*$ is positive. Let $\mathbf{x}$ be the vector obtained from $\mathbf{x}^*$ replacing $x_{ki}^{(m)*}$ by $x_{ki}^{(m)}$, where $0 \le x_{ki}^{m} < x_{ki}^{m*}$ and $\mathbf{x} \notin D$, $\mathbf{x} \in \mathop{X}_{k=1}^{N} S_k$. Then we have

$$\varphi_k(\mathbf{x}) = -K_k(x_{k1}^{(1)*}, \ldots, x_{ki}^{(m)}, \ldots, x_{ki_k}^{(M)*}) > \tag{2.4}$$
$$> -K_k(x_{k1}^{(1)*}, \ldots, x_{ki}^{(m)*}, \ldots, x_{ki_k}^{(M)*}) = \varphi_k(\mathbf{x}^*),$$

which contradicts to (1.3).

c) Consider next an oligopoly game with pay-off functions of the original game and set of simultaneous strategies S. We will now prove that if x^* is an equilibrium point of this game, then it is an equilibrium point of the original oligopoly game.

Assume first that $\mathbf{x} = (\mathbf{x}_1^*, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N^*) \in S$ (where $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{v}_N^*)$), than (1.3) is true obviously and for $\mathbf{x} \notin S$ we have

$$\varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N^*) = -K_k(\mathbf{x}_k) < \tag{2.5}$$
$$< -K_k(\mathbf{0}) = \varphi_k(\mathbf{x}_1^*, \ldots, \mathbf{0}, \ldots, \mathbf{x}_N^*) \le \varphi_k(\mathbf{x}^*),$$

because $(\mathbf{x}_1^*, \ldots, \mathbf{0}, \ldots, \mathbf{x}_N^*) \in S$.

d) Since φ_k is continuous and concave in $\mathbf{x}_k$ on the set S, the generalized version of the Nikaido – Isoda theorem implies that the game with reduced set of simultaneous strategies has at least one equilibrium point, and the previous step implies that all of these equilibrium points are equilibrium points also of the original oligopoly game.

Remark. The parts *b)* and *c)* are valied also dropping the condition according to matrix $\mathbf{A}+\mathbf{A}^T$, therefore in the general linear oligopoly game it is sufficient to consider the reduced game.

3. Reduction to quadratic programming

In this section we will assume that

E) $(L^{(1)}, \ldots, L^{(M)}) \in D$.

Then assumption *c)* implies that $S_1 \times S_2 \times \ldots \times S_N \subset D$. Let us assume further that $\mathbf{A}+\mathbf{A}^T$ is negativ semidefinite. Since φ_k is concave in $\mathbf{x}_k$, a vector $\mathbf{x}^* = (x_{11}^{(1)*}, \ldots, x_{11}^{(M)*}, \ldots, x_{Ni_N}^{(1)*}, \ldots, x_{Ni_N}^{(M)*})$ is an equilibrium point if and only if

$$\text{(3.1)} \qquad \sum_{m=1}^{M} s_k^{(m)*} a_m^{(\mu)} + \sum_{m=1}^{M} a_\mu^{(m)} s^{(m)*} + b_\mu - A_{kj}^{(\mu)} \begin{cases} \leqq 0, & \text{if } x_{kj}^{(\mu)*} = 0 \\ \geqq 0, & \text{if } x_{kj}^{(\mu)*} = L_{kj} \quad (\forall k, j, \mu) \\ = 0, & \text{if } 0 < x_{kj}^{(\mu)*} < L_{kj}^{(\mu)} \end{cases}$$

where

$$s_k^{(m)*} = \sum_{i=1}^{i_k} x_{ki}^{(m)*}, \; s^{(m)*} = \sum_{k=1}^{N} s_k^{(m)*} \; (1 \leq k \leq N, \; 1 \leq m \leq M).$$

Introduce the following slack variables:

$$\text{(3.2)} \qquad w_{kj}^{(\mu)} = L_{kj}^{(\mu)} - x_{kj}^{(\mu)} \geqq 0; \quad z_{kj}^{(\mu)} \begin{cases} = 0, & \text{if } x_{kj}^{(\mu)} > 0 \\ \geqq 0, & \text{otherwise} \end{cases};$$

$$v_{kj} \begin{cases} = 0, & \text{if } x_{kj}^{(\mu)} < L_{kj}^{(\mu)} \\ \geqq 0, & \text{otherwise} \end{cases},$$

then (3.1) is obviously equivalent to the equations

$$\text{(3.3)} \qquad \sum_{m=1}^{M} s_k^{(m)*} a_m^{(\mu)} + \sum_{m=1}^{M} a_m^{(\mu)} s^{(m)*} + b_\mu - A_{kj}^{(\mu)} + v_{kj}^{(\mu)} + z_{kj}^{(\mu)} = 0 (\forall k, j, \mu).$$

Let us now introduce the following hypermatrices

$$\mathbf{C} = (\mathbf{C}_{pq})_{p,q=1}^{N}, \quad \mathbf{B} = (\mathbf{B}_{pq})_{p,q=1}^{N},$$

where $\mathbf{C}_{pq}$ is an $i_p \times i_q$ matrix with unit elements, for $p \neq q$, $\mathbf{B}_{pq} = \mathbf{0}$, and $\mathbf{B}_{pp} = \mathbf{C}_{pp}$ $(1 \leq p \leq N,\ 1 \leq q \leq N)$.

Definition 2. Let $\mathbf{G} = (g_{ij})_{i\,j=1}^{pq}$, $\mathbf{H} = (h_{ij})_{i,j=1}^{u,v}$ two matrices with constant elements. The direct product of matrices $\mathbf{G}$ an $\mathbf{H}$ is the hypermatrix:

$$\mathbf{G} \times \mathbf{H} = \begin{pmatrix} g_{11}\mathbf{H} & g_{12}\mathbf{H} & \dots & g_{1q}\mathbf{H} \\ g_{21}\mathbf{H} & g_{22}\mathbf{H} & \dots & g_{2q}\mathbf{H} \\ \dots & \dots & \dots & \dots \\ g_{p1}\mathbf{H} & g_{p2}\mathbf{H} & \dots & g_{pq}\mathbf{H} \end{pmatrix}. \tag{3.4}$$

Simple calculation shows that equations (3.3) with complementary slackness can be rewritten in the form

$$\begin{gathered} \mathbf{P}\mathbf{x} + \mathbf{b} - \mathbf{a} - \mathbf{v} + \mathbf{z} = O \\ \mathbf{x} + \mathbf{w} = \mathbf{1} \\ \mathbf{x}^T \mathbf{z} = \mathbf{v}^T \mathbf{w} = \mathbf{v}^T \mathbf{z} = O \\ \mathbf{x}, \mathbf{v}, \mathbf{z}, \mathbf{w} \geqq O\,, \end{gathered} \tag{3.5}$$

where $\mathbf{P} = \mathbf{C} \times \mathbf{A} + \mathbf{B} \times \mathbf{A}^T$, and the components of $\mathbf{x}, \mathbf{a}, \mathbf{v}, \mathbf{z}, \mathbf{w}, \mathbf{1}$ are

$$x_{ki}^{(m)},\ A_{ki}^{(m)},\ v_{ki}^{(m)},\ z_{ki}^{(m)},\ w_{ki}^{(m)},\ L_{ki}^{(m)},$$

respectively, furthermore

$$\mathbf{b} = (b_1, \dots, b_M, \dots, b_1, \dots, b_M)\,.$$

Observe that the form of equations (3.5) is the same as the one in the paper of Manas ([4]).

Thus we proved the following theorem.

Theorem 2. Assuming A), B), C) , D), E) any equilibrium point of the linear oligopoly game satisfy the equations (3.5). In the further special case when $\mathbf{A} + \mathbf{A}^T$ is negativ semidefinite, each solution of equations (3.5) is also equilibrium point.

Next we will prove two lemmas.

Lemma 1. The matrix $\mathbf{B} + \mathbf{C}$ is positive semidefinit and in the case of $i_1 = i_2 = \dots = i_N = 1$ the matrix $\mathbf{B} + \mathbf{C}$ is positive definite.

Proof. Let $\mathbf{u} = (u_{11}, \dots, u_{1i_1}, \dots, u_{N1}, \dots, u_{Ni_N})$ be any vector. Then

$$\mathbf{u}^T (\mathbf{B} + \mathbf{C})\, \mathbf{u} = \left(\sum_{k=1}^{N} \sum_{i=1}^{i_k} u_{ki} \right)^2 + \sum_{k=1}^{N} \left(\sum_{i=1}^{i_k} u_{ki} \right)^2 \geqq 0, \tag{3.6}$$

and for $i_k = 1$ $(k = 1, 2, \dots, N)$, $u^T (\mathbf{B} + \mathbf{C})\ u = 0$ if and only if $u_{k1} = 0$ $(1 \leq k \leq N)$.

Lemma 2. Let $\mathbf{U}$ and $\mathbf{V}$ be positive definite matrices.
Then $\mathbf{U} \times \mathbf{V}$ is also positive definite.

Proof. Let $\lambda_1, \dots, \lambda_p$ and $\mu_1, \dots, \mu_q$ be the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$, respectively.

Then the theorem of Stephanos and Egerváry (see [10]) implies that the eigenvalues of $\mathbf{U}\times\mathbf{V}$ are the values $\lambda_i\,\mu_j$ $(1\le i\le p,\ 1\le j\le q)$, consequently the eigenvalues of $\mathbf{U}\times\mathbf{V}$ are positive numbers. The matrix $\mathbf{U}\times\mathbf{V}$ is obviously symmetric, which implies the assertion.

The following theorem will be proved next.

Theorem 3. Assume that A), B), C), D), E) are satisfied, and matrix $\mathbf{A}$ is symmetric and negative semidefinite. Then all of the equilibrium points of the linear oligopoly game are optimial solutions of the quadratic programming problem:

$$\frac{\mathbf{0}\le\mathbf{x}\le\mathbf{1}}{\frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x}+(\mathbf{b}-\mathbf{a})^T\mathbf{x}\to\max}. \tag{3.7}$$

Proof. Since the assumptions of theorem 1. are satisfied, the oligopoly game has of least one equilibrium point, which, according to theorem 2, fulfils (3.5). If we do not take account the condition $\mathbf{v}^T\mathbf{z}=O$ in (3.5), we can easily verify that remaining relations give the Kuhn-Tucker conditions for the concave quadratic programming problem (3.7).

Let us now consider the special case, if $i_1=i_2=\ldots=i_N=1$.

Theorem 4. If assumptions A), B), C), D), E) are satisfied and $i_k=1$ $(1\le k\le N)$, furthermore matrix $\mathbf{A}$ is symmetric and negative definite, then the linear oligopoly game has a unique equilibrium point, which can be obtained as the unique optimal solution of the quadratic programming problem (3.7).

Proof. The assertion of the theorem follows immidiately from Theorem 3 and from the fact, that the quadratic programming problem (3.7) is strictly concave.

REFERENCES

[1] *Aoki, M.: Introduction to Optimization Techniques. Fundamentals and Applications of Nonlinear Programming.* The Macmillan Comp., N.Y. 1971.
[2] *Burger, E.: Einführung in the Theorie der Spiele.* Walter de Gruyter and Co., Berlin, 1959.
[3] *Frank, Jr. C. R., Quandt, R. E.:* On the Existence of Cournot Equilibrium. International Economic Review, Vol. 4., No. 1., 1973. pp. 92–96.
[4] *Manas, M.:* A Linear Oligopoly Game. Econometrica, Vol. 40, 1972, pp. 917–922.
[5] *Opitz, O.:* Spieltheoretische Aussagen in Oligopolproblem. Zeitschrift für Nationalökonomie, Vol. 30, 1970, pp. 475–482.
[6] *Rosen, J. B.:* Existence and Uniqueness of Equilibrium Points for Concave n-Person Games. Econometrica, Vol. 33, 1965, pp. 520–534.
[7] *Szidarovszky, F.:* On the Oligopoly Game. K. Marx Univ. of Economics, Budapest, Tech. rep. 1970–1.
[8] *Szidarovszky, F.:* On the Group-Equilibrium Problem of the Oligopoly Game. Thesis presented for the degree "Cand. for Math.", Budapest, 1974.
[9] *Szidarovszky, F.:* On the Concave Oligopoly Problem. K. Marx Univ. of Economics, Budapest, Tech. rep. 1976–1.
[10] *Szidarovszky, F.: Numerical Methods of Linear Algebra* (in Hungarian), Tankönyvkiadó, Budapest, 1972.

University of Agriculture Dept. of Computer Science
1113 Budapest, Villányi út 29–35.