ON UNIQUE EQUILIBRIUM POINTS OF CONCAVE n-PERSON GAMES

by

FERENC SZIDAROVSZKY

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In paper [1] J. B. Rosen gave sufficient conditions for the uniqueness of the equilibrium point of a general n-person game.

R_i denotes the set of the strategies of the ith player. Assume that for $i = 1, 2, \dots, n$

$$R_i = \{\mathbf{x}_i \mid \mathbf{x}_i \in E^{m_i}, \mathbf{h}_i(\mathbf{x}_i) \geq 0\} \tag{1}$$

where each component $h_{ij}(\mathbf{x}_i)$ $(j = 1, 2, \dots, k_i)$ of $\mathbf{h}_i(\mathbf{x}_i)$ $(i = 1, 2, \dots, n)$ is a concave function of $\mathbf{x}_i$. It is assumed, that R_i is nonvoid and bounded. We will also assume that the set R_i contains a point that is strictly interior to every nonlinear constraint.

So R_i is a convex, closed set $(1 \leq i \leq n)$.

Let R be the set of the strategies of all players, then $R \subset E^m$, where

$$m = \sum_{i=1}^{n} m_i .$$

Assume that

$$R = R_1 \times R_2 \times \dots \times R_n .$$

We assume that the pay-off function of the ith player is the following:

$$\varphi_i(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{i_{k1} + \dots + i_{km} = k} a^{(i)}_{i_{k1} i_{k2} \dots i_{km}} x_1^{i_{k1}} x_2^{i_{k2}} \dots x_m^{i_{km}} .$$

For a fixed vector $\mathbf{r} \in E^n$, $\mathbf{r} > \mathbf{0}$, $\mathbf{r} = (r_1, \dots, r_n)$ let

$$\mathbf{g}(\mathbf{x}, \mathbf{r}) = \begin{bmatrix} r_1 \Delta_1 \varphi_1(\mathbf{x}) \\ r_2 \Delta_2 \varphi_2(\mathbf{x}) \\ \dots\dots\dots \\ r_n \Delta_n \varphi_n(\mathbf{x}) \end{bmatrix} .$$

where $\Delta_i \varphi_i(\mathbf{x})$ denotes the gradient with respect to $\mathbf{x}_i$ of $\varphi_i(\mathbf{x})$. Let $[G(\mathbf{x},\mathbf{r})]$ be the Jacobian-matrix of $\mathbf{g}(\mathbf{x},\mathbf{r})$, that the is, jth column of $[G(\mathbf{x},\mathbf{x})]$ is $\partial \mathbf{g}(\mathbf{x},\mathbf{r})\backslash \partial x_j$ ($j = 1, 2, \ldots, m$).

It is easy to prove, that

$$[G(\mathbf{x},\mathbf{r})] = D[C(\mathbf{x})]$$

where

$$D = \operatorname{diag}\{r_i\}$$

and the jth element of the ith row of the matrix $[C(\mathbf{x})]$ is the following

$$\frac{\partial^2 \varphi_l(x)}{\partial x_i \partial x_j}$$

where

$$1 \leq i, j \leq m, \quad \sum_{t=1}^{l-1} m_t < i \leqq \sum_{t=1}^{l} m_t .$$

If the matrix $[G(\mathbf{x}, \mathbf{r}) + G'(\mathbf{x}, \mathbf{r})]$ is negative definite for some $\mathbf{r} > \mathbf{0}$, then the game defined above has a unique equilibrium point [1].

Consider the approximation of this game with the same sets of strategies and the pay-off functions

$$\bar{\varphi}_i(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{i_{k1}+\ldots+i_{km}=k} \bar{a}^{(i)}_{i_{k1} i_{k2} \ldots i_{km}} x_1^{i_{k1}} x_2^{i_{k2}} \ldots x_m^{i_{km}} .$$

Let the matrix $[\bar{G}(\mathbf{x}, \mathbf{r}) + \bar{G}'(\mathbf{x}, \mathbf{r})]$ be the corresponding matrix to $[G(\mathbf{x}, \mathbf{r}) + G'(\mathbf{x}, \mathbf{r})]$ for this game.

Assume that for $k = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$

(2)
$$\left| a^{(i)}_{i_{k1} i_{k2} \ldots i_{km}} - \bar{a}^{(i)}_{i_{k1} i_{k2} \ldots i_{km}} \right| \leq \varepsilon_k .$$

Let

$$\alpha_k(x_1, \ldots, x_m) = \sum_{i_{k1}+\ldots+i_{km}=k} x_1^{i_{k1}} x_2^{i_{k2}} \ldots x_m^{i_{km}}$$

$$\beta_{lij}(x_1, \ldots, x_m) = \frac{\partial^2 \alpha_l(|x_1|, \ldots, |x_m|)}{\partial x_i \partial x_j} \quad (1 \leq i, j \leq m)$$

and

$$\delta(x_1, \ldots, x_m) = \sum_{k=0}^{\infty} \varepsilon_k \alpha_k(x_1, \ldots, x_m) .$$

Assume that the series $\varphi_i(\mathbf{x})$, $\bar{\varphi}_i(\mathbf{x})$ ($1 \leq i \leq n$), $\delta(\mathbf{x})$ are absolute convergent and can be differentiated twice by terms for all $\mathbf{x} \in R$.

If the matrices $[G(\mathbf{x}, \mathbf{r}) + G'(\mathbf{x}, \mathbf{r})]$ are negative definite for all $\mathbf{x} \in R$ and fixed $\mathbf{r} > 0$, then there exists a positive number T such that the eigenvalues of the matrices $[G(\mathbf{x}, \mathbf{r}) + G'(\mathbf{x}, \mathbf{r})]$ are less than $-T$. Let

$$r = \max_{1 \leq i \leq n} \{r_i\}.$$

The following theorem is true.

Theorem. If the conditions above hold and

$$\max_{1 \leq i, j \leq m} \max_{\mathbf{x} \in R} \sum_{k=0}^{\infty} \varepsilon_k \beta_{kij}(\mathbf{x}) \leq \frac{T}{2mr} \tag{3}$$

then the game with the sets of strategies R_i and pay-off functions $\bar{\varphi}_i(\mathbf{x})$ has a unique equilibrium point.

In the proof we will use a theorem about the variation of the spectrum of symmertic matrices. Its proof can be found e.g. in [2] and [3]. The theorem is as follows:

Let $B = (b_{ij})_{i,j=1}^m$ and $\bar{B} = (\bar{b}_{ij})_{i,j=1}^m$ be two symmetric matrices. Assume that for $i, j = 1, 2, \ldots, m$

$$|b_{ij} - \bar{b}_{ij}| \leq \varepsilon,$$

then the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ of B and $\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \ldots \leq \bar{\lambda}_m$ f $\bar{B}$ satisfy the inequalities

$$|\lambda_i - \bar{\lambda}_i| \leq m\,\varepsilon \quad (1 \leq i \leq m).$$

Proof of the theorem.

Simple calculation shows that the moduli of the differences of the corresponding elements of the matrices $[G(\mathbf{x}, \mathbf{r}) + G'(\mathbf{x}, \mathbf{r})]$ and $[\bar{G}(\mathbf{x}, \mathbf{r}) + \bar{G}'(\mathbf{x}, \mathbf{r})]$ are not greater than

$$2r \sum_{k=0}^{\infty} \varepsilon_k \beta_{kij}(\mathbf{x}) \leq 2r \max_{1 \leq i, j \leq m} \max_{\mathbf{x} \in R} \sum_{k=0}^{\infty} \varepsilon_k \beta_{kij}(\mathbf{x}).$$

The matrix $[\bar{G}(\mathbf{x}, \mathbf{r}) + \bar{G}'(\mathbf{x}, \mathbf{r})]$ is also negative definite and so the approximating game has a unique equilibrium point, if

$$m \cdot 2r \max_{1 \leq i, j \leq m} \max_{\mathbf{x} \in R} \sum_{k=0}^{\infty} \varepsilon_k \beta_{kij}(\mathbf{x}) \leq T \tag{4}$$

in consequence of the definition of T and the theorem mentioned above, and (4) is equivalent to (3).

Thus the theorem is proved.

Remark 1. In the case, when $R \subset R_1 \times R_2 \times \ldots \times R_n$ J. B. Rosen gave sufficient condition for the uniqueness of a so called normalized equilibrium point. The theorem proved above can be adapted also for this case without any difficulty.

Remark 2. If the pay-off functions have the form

$$\varphi_i(\mathbf{x}) = \sum_{j=1}^{n} [\mathbf{e}'_{ij'} + \mathbf{x}'_i c_{ij}]\, \mathbf{x}_j \quad (i = 1, 2, \ldots, n)$$

– where $\mathbf{e}_{ij}$ is a constant vector in E^{m_i} and C_{ij} is an $m_i \times m_j$ constant matrix – the above result gives a better estimation than the one in [4].

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University of Agriculture Dept. of Computer Science
1113 Budapest, Villányi út 29–35.