SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS, III.

APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATION $y' = f(x, y)$ WITH SPLINE FUNCTIONS

By

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Abstract. A method for obtaining a spline function approximation for the solution of the non-linear differential equation $y' = f(x, y)$, where $f \in C^r$, is presented. The existence, uniqueness and convergence of the approximate solution are investigated.

1. Introduction and description of the method

The problem of approximating the solution of the non-linear first order differential equation with spline functions has been solved by Frank R. Loscalzo and Thomas D. Talbot [2], [3]. The method described by them produces convergent quadratic and cubic spline approximations, but when applied with higher degree splines, their method is divergent as it is observed from their following theorem.

Theorem 1.1: (See [3] page 444). The solutions $S_m(x)$ are divergent as $h \to 0$ for $m \geqq 4$.

Their method is applicable only if $f \in C^2$, C^3 (while the method does not succeed for $f \in C^0$ and C^1) and under the restriction that the step h must be smaller than $2/L$ and $3/L$ consequently, where L is the Lipschitz constant satisfied by f. The main convergence theorems in their method are as follows.

Theorem 1.2: (Loscalzo and Talbot). If $f(x, y) \in C^2$ in T, then there exists a constant K such that for all $h < 2/L$,

$$|S_2(x) - y(x)| < Kh^2, \quad |S_2'(x) - y'(x)| < Kh^2, \quad |S_2''(x) - y''(x)| < Kh,$$

if $x \in [0, b]$, provided $S_2''(x_k)$ is given by:

$$S^{(m)}(x_k) = \frac{1}{2}\left[S^{(m)}\left(x_k - \frac{1}{2}h\right) + S^{(m)}\left(x_k + \frac{1}{2}h\right)\right], \quad k = 1, \ldots, (n-1)$$

with $m = 2$.

Theorem 1.3: (Loscalzo and Talbot). If $f(x,y) \in C^3$ in T, then there exists a constant K such that for all $h < 3/L$,

$$|S_3(x) - y(x)| < Kh^4, \quad |S_3'(x) - y'(x)| < Kh^3,$$
$$|S_3''(x) - y''(x)| < Kh^2, \quad |S_3'''(x) - y'''(x)| < Kh,$$

if $x \in [0, b]$, provided $S_3'''(x_k)$ is given by

$$S^{(m)}(x_k) = \frac{1}{2}\left[S^{(m)}\left(x_k - \frac{1}{2}h\right) + S^{(m)}\left(x_k + \frac{1}{2}h\right)\right]$$

with $m = 3$.

More details about these theorems are found in [3].

In this paper, following the same method presented in [4] and [5] we are going to approximate the solution of the problem $y' = f(x, y)$ where $f \in C^r$ and r may be any finite positive integer or zero. Also by this method we avoid the restrictions in the above theorems and we construct spline functions of degree $\geqq 3$ which approximate the solution of the problem and converge faster to the exact solution.

For this purpose, consider the Cauchy problem in the nonlinear ordinary differential equation

$$y'(x) = f(x, y(x)) \quad \text{with} \quad y(0) = y_0, \tag{1.1}$$

where $f \in C^r$ $([0, b]\, xR)$ and r is finite positive integer or zero.

If $S(x)$ is the spline function approximating the solution of (1.1), it satisfies

$$S(x) \in C^{r+1}[0, b] \tag{1.2}$$

$$S(x) \in \pi_m \quad \text{in each subinterval} \quad [x_k, x_{k+1}], \quad k = 0, 1, \ldots, (n-1) \tag{1.3}$$

where we define the knots by

$$0 = x_0 < x_1 < \ldots < x_n = b \tag{1.4}$$

and in our case we shall deal with equal subintervals and in this paper we denote

$$x_{k+1} - x_k = h, \quad k = 0, 1, \ldots, (n-1). \tag{1.5}$$

Here π_m denotes the set of all polynomials of degree $\leqq m$, and $m = 2r+3$.

Furthermore, in what follows, $c_0, c_1, c_2, \ldots$ shall denote constants independent of h and consequently independent of n.

In this paper too, as in our previous papers, we assume that (1.1) represents a single scalar equation, but nearly all the numerical and theoretical considerations in this paper carry over to systems of first order equations, where (1.1) could be treated in vector form. Moreover we shall use the Lipschitz condition on f to guarantee the existence of a unique analytical solution of (1.1).

Following the same steps as in [4] and [5], our method to approximate the solution of (1.1) will be divided into two main approximation processes, the first of which is to obtain, numerically, the approximate values $\bar{y}_k, \bar{y}'_k, \ldots, \bar{y}_k^{(r+1)}$, which are the approximate values of $y(x), y'(x), \ldots, y^{(r+1)}(x)$ at $x = x_k$, where $k = 0, 1, \ldots, n$. Here $y(x)$ is the exact solution of (1.1). The second approximation process contains the construction of the spline function which approximates the solution and also contains theorems regarding the convergence of this function to the exact solution.

Thus we start with the following.

2. The first approximation process

This chapter contains some assumptions concerning the functions f and a method for obtaining the approximate values $\bar{y}_k, \bar{y}'_k, \ldots, \bar{y}_k^{(r+1)}$, where $k = 0, 1, \ldots, n$, and also we discuss in this chapter the convergence of these values to the exact ones.

2.1 Assumptions and procedure of the method. If

$$T = \{(x, y) \mid 0 \leqq x \leqq b\},$$

then we assume that f satisfies the following conditions

$$f(x, y(x)) \in C^r \quad \text{in} \tag{2.1.1}$$

$$D: |x - x_0| < \alpha, \quad |y - y_0| < \beta \quad \text{where} \quad T \subset D$$

together with the Lipschitz conditions

$$|f^{(q)}(x, y_1) - f^{(q)}(x, y_2)| \leqq L\, |y_1 - y_2| \tag{2.1.2}$$

for all (x, y_1) and (x, y_2) in D, where L is some Lipschitz constant and $q = 0, 1, \ldots, r$.

Here $f^{(q)}(x, y)$ denotes the q-th total derivative of $f(x, y)$ w.r.t x using the following algorithm:

$$y^{(n)}(x) = f^{(n-1)}(x, y(x)), \quad n = 1, 2, \ldots, r+1,$$

where the functions $f^{(m)}$ can be calculated by means of the recurrence relation

$$f^{(0)} = f, \quad f^{(m+1)} = f_x^{(m)} + f_y^{(m)} f, \quad m = 0, 1, \ldots .$$

We also assume that $f^{(r)}(x, y(x))$, as a function of x only in the case of the existence of a unique solution $y(x)$, has a modulus of continuity $\omega(f^{(r)}, h) = \omega_r(h)$.

Let $y(x)$ be the exact solution of (1.1) with the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$. Then by integrating (1.1) from x_k to x, where $x_k \leqq x \leqq x_{k+1}$ and $k = 0, 1, \ldots, (n-1)$, we get

$$y(x) = y_k + \int_{x_k}^{x} f(t, y(t))\, dt \tag{2.1.3}$$

and by putting $x = x_{k+1}$ we get

(2.1.4) $$y(x_{k+1}) = y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} f(t, y(t))\, dt$$

and we assume that this may be approximated by the following formula:

(2.1.5) $$\bar{y}_{k+1} = \bar{y}_k + \int_{x_k}^{x_{k+1}} f(t, y_k^*(t))\, dt,$$

where $y_k^*(t)$ is defined by the expansion

(2.1.6) $$y_k^*(t) = \sum_{j=0}^{r+1} (t-x_k)^j \frac{\bar{y}_k^{(j)}}{j!}, \quad x_k \leqq t \leqq x_{k+1}$$

and this expansion corresponds to the Taylor expansion

(2.1.7) $$y(t) = \sum_{j=0}^{r} \frac{y_k^{(j)}}{j!}(t-x_k)^j + \frac{y^{(r+1)}(\xi_k)}{(r+1)!}(t-x_k)^{r+1},$$
$$x_k < \xi_k < x_{k+1}.$$

Finally, knowing that the derivatives of $f(x, y)$ are also functions of x and y we can define

(2.1.8) $$y_{k+1}^{(q+1)} = f^{(q)}(x_{k+1}, y_{k+1})$$

and

(2.1.9) $$\bar{y}_{k+1}^{(q+1)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}),$$

where $q = 0, 1, \ldots, r$ and $k = 0, 1, \ldots, (n-1)$.

We can start our calculations by using the substitutions $\bar{y}_0 = y_0$, $\bar{y}_0' = y_0', \ldots, \bar{y}_0^{(r+1)} = y_0^{(r+1)}$ and so we can proceed to the convergence theorems as the following.

2.2 General convergence processes. In this paragraph we prove theorems concerning the convergence of the approximate values $\bar{y}_{k+1}, \bar{y}_{k+1}', \ldots, \bar{y}_{k+1}^{(r+1)}$ to the exact values $y(x_{k+1}), y'(x_{k+1}), \ldots, y^{(r+1)}(x_{k+1})$, where $k = 0, 1, \ldots, (n-1)$. Before proving these theorems we start with the following lemma

Lemma 2.2.1 The inequality

$$|y_{k+1} - \bar{y}_{k+1}| \leqq |y_k - \bar{y}_k| (1 + c_0 h) + c_1 \omega_r(h) h^{r+2}$$

is true for all $k = 0, 1, \ldots, (n-1)$.

Proof. Using equations (2.1.4) and (2.1.5) together with the Lipschitz condition (2.1.2) we get

$$|y_{k+1} - \bar{y}_{k+1}| \leqq |y_k - \bar{y}_k| + L \int_{x_k}^{x_{k+1}} |y(t) - y_k^*(t)|\, dt$$

and by using equations (2.1.6) and (2.1.7) for $y_k^*(t)$ and $y(t)$ respectively this becomes

$$\leqq |y_k - \bar{y}_k| + $$

$$+ L \int_{x_k}^{x_{k+1}} \left| \sum_{j=0}^{r} \frac{y_k^{(j)}}{j!} (t-x_k)^j + \frac{y^{(r+1)}(\xi_k)}{(r+1)!} (t-x_k)^{r-1} - \sum_{j=0}^{r+1} \frac{\bar{y}_k^{(j)}}{j!} (t-x_k)^j \right| dt$$

$$\leqq |y_k - \bar{y}_k| + L \sum_{j=0}^{r+1} \frac{|y_k^{(j)} - \bar{y}_k^{(j)}|}{j!} \int_{x_k}^{x_{k+1}} (t-x_k)^j \, dt +$$

$$+ L \frac{\omega_r(h)}{(r+1)!} \int_{x_k}^{x_{k+1}} (t-x_k)^{r+1} \, dt$$

$$\leqq |y_k - \bar{y}_k| + L \sum_{j=0}^{r+1} \frac{h^{j+1}}{(j+1)!} |y_k - \bar{y}_k^{(j)}| + L \frac{h^{r+2}}{(r+2)!} \omega_r(h)$$

$$= |y_k - \bar{y}_k| + Lh |y_k - \bar{y}_k| + L \sum_{q=0}^{r} \frac{|f^{(q)}(x_k, y_k) - f^{(q)}(x_k, \bar{y}_k)|}{(q+2)!} h^{q+2} +$$

$$+ L \frac{h^{r+2}}{(r+2)!} \omega_r(h)$$

applying the Lipschitz condition (2.1.2) once more this will be

$$\leqq |y_k - \bar{y}_k| + Lh |y_k - \bar{y}_k| + L^2 |y_k - \bar{y}_k| \sum_{q=0}^{r} \frac{h^{q+2}}{(r+2)!} \omega_r(h) h^{r+2}$$

$$\leqq |y_k - \bar{y}_k| (1 + c_0 h) + c_1 \omega_r(h) h^{r+2}$$

which is the required result.

Theorem 2.2.1 The convergence of the approximate value $\bar{y}_{k+1}$ given by the formula (2.1.5) to the value of the exact solution of (1.1) at x_{k+1} is given by the inequality

$$|y_{k+1} - \bar{y}_{k+1}| \leqq c_3 \omega_r(h) h^{r+1}$$

which holds for all $k = 0, 1, \ldots, (n-1)$.

Proof. By successive substitutions for $|y_i - \bar{y}_i|$ from the lemma 2.2.1, where i takes the values $k+1, k, k-1, \ldots, 3, 2, 1$ respectively we get

$$\begin{array}{lll}
|y_{k+1} - \bar{y}_{k+1}| & \leqq |y_k - \bar{y}_k| (1 + c_0 h) & + c_1 \omega_r(h) h^{r+2} \\
|y_k - \bar{y}_k| (1 + c_0 h) & \leqq |y_{k-1} - \bar{y}_{k-1}| (1 + c_0 h)^2 & + c_1 \omega_r(h) h^{r+2} (1 + c_0 h) \\
|y_{k-1} - \bar{y}_{k-1}| (1 + c_0 h)^2 & \leqq |y_{k-2} - \bar{y}_{k-2}| (1 + c_0 h)^3 & + c_1 \omega_r(h) h^{r+2} (1 + c_0 h)^2 \\
\cdot & \leqq \quad \cdot & + \quad \cdot \\
\cdot & \leqq \quad \cdot & + \quad \cdot \\
\cdot & \leqq \quad \cdot & + \quad \cdot \\
|y_1 - \bar{y}_1| (1 + c_0 h)^k & \leqq |y_0 - \bar{y}_0| (1 + c_0 h)^{k+1} & + c_1 \omega_r(h) h^{r+2} (1 + c_0 h)^k
\end{array}$$

and the result will be

$$|y_{k+1}-\bar{y}_{k+1}| \leqq |y_0-\bar{y}_0|\,(1+c_0 h)^{k+1}+c_1\,\omega_r(h)\,h^{r+2}\sum_{j=0}^{k}(1+c_0 h)^j$$

and by taking into consideration that $y_0=\bar{y}_0$ we get

$$\begin{aligned}|y_{k+1}-\bar{y}_{k+1}| &\leqq c_1\,\omega_r(h)\,h^{r+2}\sum_{j=0}^{k}(1+c_0 h)^j\\ &= c_1\,\omega_r(h)\,h^{r+2}\,\frac{(1+c_0 h)^{k+1}-1}{c_0 h}\\ &= \frac{c_1}{c_0}\,\omega_r(h)\,h^{r+1}\{(1+c_0 h)^{k+1}-1\}\end{aligned}$$

and by knowing that

$$(1+c_0 h)^{k+1}=\left(1+\frac{bc_0}{n}\right)^{k+1}\leqq\left(1+\frac{bc_0}{n}\right)^{n}\leqq e^{bc_0}=\text{constant}$$

we get the required result which completes the proof.

Theorem 2.2.2 The error of $\bar{y}_{k+1}^{(q+1)}$ given by the formula (2.1.9) is estimated by the inequality

$$|y_{k+1}^{(q+1)}-\bar{y}_{k+1}^{(q+1)}|\leqq c_4\,\omega_r(h)\,h^{r+1},\quad q=0,1,\ldots,r$$

which holds for all $k=0,1,\ldots,(n-1)$.

Proof. From the equations (2.1.8) and (2.1.9) we get

$$|y_{k+1}^{(q+1)}-\bar{y}_{k+1}^{(q+1)}| = |f^{(q)}(x_{k+1},y_{k+1})-f^{(q)}(x_{k+1},\bar{y}_{k+1})|$$

applying the Lipschitz condition (2.1.2) this will be

$$\leqq L\,|y_{k+1}-\bar{y}_{k+1}|$$

and by theorem 2.2.1 this becomes

$$\begin{aligned}&\leqq c_3\,L\,\omega_r(h)\,h^{r+1}\\ &\leqq c_4\,\omega_r(h)\,h^{r+1}\end{aligned}$$

and thus the proof is complete.

3. The second approximation process

We have obtained, as we have seen before, the sets of the approximate values

$$\bar{Y}^{(q)}:\bar{y}_0^{(q)},\bar{y}_1^{(q)},\ldots,\bar{y}_n^{(q)}\quad q=0,1,\ldots,r+1$$

which are corresponding to

$$Y^{(q)}: y_0^{(q)}, y_1^{(q)}, \ldots, y_n^{(q)} \quad q = 0, 1, \ldots, r+1$$

respectively. In this chapter and on the bases of those sets of approximate values we are going to construct a spline function $S_\Delta(x)$ interpolated to the set $\bar{Y}$ on the mesh Δ and approximating the solution of (1.1) and also we shall discuss the convergence of this function to $y(x)$.

3.1 The construction of the spline function. In this paragraph we introduce the spline function which approximates the solution of our differential equation and so we introduce the following theorem.

Theorem 3.1 For a given mesh of points

$$\Delta: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b, \qquad x_{k+1} - x_k = h$$

and for given sets of values

$$\bar{Y}^{(q)}: \bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \ldots, \bar{y}_n^{(q)}, \quad q = 0, 1, \ldots, r+1$$

there is a unique spline function $S_\Delta(x)$ interpolated on the mesh Δ to the set $\bar{Y}$ and satisfying the following conditions

$$S_\Delta(\bar{Y}, x) = S_\Delta(x) \in C^{r+1}[0, b] \tag{3.1.1}$$

$$S_k^{(q)}(x_k) = \bar{y}_k^{(q)}, \; S_{n-1}^{(q)}(x_n) = \bar{y}_n^{(q)} \quad q = 0, 1, \ldots, r+1, \quad k = 0, 1, \ldots, (n-1). \tag{3.1.2}$$

For $x_k \leqq x \leqq x_{k+1}$ and $k = 0, 1, \ldots, (n-1)$

$$S_\Delta(x) = S_k(x) = \sum_{j=0}^{r+1} \frac{\bar{y}_k^{(j)}}{j!}(x - x_k)^j + \sum_{p=1}^{r+2} a_p^{(k)}(x - x_k)^{p+r+1}. \tag{3.1.3}$$

Proof. From the continuity condition (3.1.1) and by using (3.1.1) for $x = x_{k+1}$ we get

$$S_k^{(t)}(x_{k+1}) = S_{k+1}^{(t)}(x_{k+1}) = \bar{y}_{k+1}^{(t)}, \quad k = 0, 1, \ldots, (n-1), \quad t = 0, 1, \ldots, r+1. \tag{3.1.4}$$

Substituting from (3.1.4) in (3.1.3) we get the system of equations

$$\sum_{p=1}^{r+2} t! \binom{p+r+1}{t} a_p^{(k)} h^{p-1} = h^{t-r-2}\left(\bar{y}_{k+1}^{(t)} - \sum_{j=0}^{r+1-t} \frac{\bar{y}_k^{(j+t)}}{j!} h^j\right) = F_t^{(k)}, \tag{3.1.5}$$

where $t = 0, 1, \ldots, r+1$ and this system of equations (3.1.5) has a unique solution for the unknowns $a_p^{(k)}, (p = 1, 2, \ldots, r+2)$, since its determinant is

$$D_r =$$

$$= \begin{vmatrix} 1 & h & \dots & h^{p-1} & \dots & h^{r+1} \\ \binom{r+2}{1} 1! & \binom{r+3}{1} 1!\, h & \dots & \binom{r+1+p}{1} 1!\, h^{p-1} & \dots & \binom{2r+3}{1} 1!\, h^{r+1} \\ \binom{r+2}{2} 2! & \binom{r+3}{2} 2!\, h & \dots & \binom{r+1+p}{2} 1!\, h^{p-1} & \dots & \binom{2r+3}{2} 2!\, h^{r+1} \\ \cdot & \cdot & & \cdot & & \\ & & & & \dots & \\ \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ \binom{r+2}{r+1} (r+1)! & \binom{r+3}{r+1} (r+1)!\, h & \dots & \binom{r+1+p}{r+1} (r+1)!\, h^{p-1} & \dots & \binom{2r+3}{r+1} (r+1)!\, h^{r+1} \end{vmatrix} =$$

$$= h \frac{1}{2} (r+1)(r+2) \prod_{t=0}^{r+1} t!,$$

which is different from zero for $h>0$. If we replace the p^{th} column in D_r by the column $(F_0^{(k)}, F_1^{(k)}, \dots, F_{r+1}^{(k)})$ and denote the determinant results by D_r^p, then the solution of the system (3.1.5) will be

$$a_p^{(k)} = \frac{D_r^p}{D_r}, \quad p = 1, 2, \dots, r+2 \tag{3.1.6}$$

and after factorizing D_r^p in terms of $F_0^{(k)}, F_1^{(k)}, \dots, F_{r+1}^{(k)}$ the solution (3.1.6) will take the form

$$a_p^{(k)} = \frac{1}{h^{p-1}} \sum_{i=0}^{r+1} c_{pi} F_i^{(k)}, \quad p = 1, 2, \dots, r+2 \tag{3.1.7}$$

and this solution, as we have said before, is unique. The uniqueness of this solution guarantees the uniqueness of the spline function $S_\Delta(x)$ and consequently the existence of such a function and thus the theorem is proved.

3.2 Convergence of the spline function to the solution. In this paragraph we prove the essential theorem concerned with the convergence of our spline function constructed in theorem 3.1 to the exact solution of (1.1), and also we prove that this function satisfies this differential equation as $n \to \infty$.

Theorem 3.2.1 Let $y(x)$ be the solution of the equation (1.1) and let $f \in C^r([0, b] xR)$. If $S_\Delta(x)$ is the spline function constructed in theorem 3.1, then there exists a constant E independent of h such that

$$|y^{(q)}(x) - S_\Delta^{(q)}(x)| \leqq E\, \omega_r(h)\, h^{r+1-q}$$

for all $x \in [0, b]$ and $q = 0, 1, \dots, r+1$.

For the proof of this theorem we are in need to prove the following lemma.

Lemma 3.2.1 The following inequalities are true

$$|a_p^{(k)}| \leqq \frac{A_p}{h^p}\,\omega_r(h), \quad p = 1, 2, \ldots, r+2,$$

where A_p are constants independent of h.

Proof of the lemma. For the proof of this lemma we deduce at first some inequalities concerning the absolute value of $F_t^{(k)}$ $(t = 0, 1, \ldots, r+1)$. They are calculated as follows.

From (3.1.5) we have

$$|F_t^{(k)}| = h^{t-r-2}\left|\bar{y}_{k+1}^{(\ )} - \sum_{j=0}^{r+1-t} \frac{\bar{y}_k^{(j+t)}}{j!}\,h^j\right|$$

and if we define the Taylor expansion of $y^{(t)}(x)$ for $x_k \leqq x \leqq x_{k+1}$ to be

$$y^{(t)}(x) = \sum_{j=0}^{r-t} \frac{y_k^{(j+t)}}{j!}(x-x_k)^j + \frac{y^{(r+1)}(\xi_{kt})}{(r+1-t)!}(x-x_k)^{r+1-t} \tag{3.2.1}$$

and $x_k < \xi_{kt} < x_{k+1}$, then we get for $x = x_{k+1}$

$$y_{k+1}^{(t)} = \sum_{j=0}^{r-t} \frac{y_k^{(j+t)}}{j!}\,h^j + \frac{y^{(r+1)}(\xi_{kt})}{(r+1-t)!}\,h^{r+1-t}, \tag{3.2.2}$$

where $t = 0, 1, \ldots, r+1$.

Using the last identity (3.2.2) together with (3.1.5) we get for $t = 0, 1, \ldots, r+1$

$$|F_t^{(k)}| \leqq h^{t-r-2}\left\{|y_{k+1}^{(t)} - \bar{y}_{k+1}^{(t)}| + \sum_{j=0}^{r-t} \frac{|y_k^{(j+t)} - \bar{y}_k^{(j+t)}|}{j!}\,h^j + \right.$$
$$\left. + \frac{|y^{(r+1)}(\xi_{kt}) - \bar{y}_k^{(r+1)}|}{(r+1-t)!}\,h^{r+1-t}\right\}.$$

Using theorems 2.2.1, 2.2.2 together with the definition of the modulus of continuity this will be

$$|F_t^{(k)}| \leqq h^{t-r+2}\{c_t^*\,\omega_r(h)\,h^{r+1-t}\},$$

where c_t^* $(t = 0, 1, \ldots, r+1)$ are constants independent of h and so we have the result that

$$|F_t^{(k)}| \leqq c_t^*\,\frac{\omega_r(h)}{h}, \quad t = 0, 1, \ldots, r+1. \tag{3.2.3}$$

Now after obtaining the last inequality, we go on to prove the lemma 3.2.1, and for this purpose we combine equations (3.2.3) with (3.1,7) to get

$$|a_p^{(k)}| \leqq \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_{pi}\,c_t^*\,\frac{\omega_r(h)}{h}$$
$$\leqq A_p\,\frac{\omega_r(h)}{h^p},$$

where A_p $(p = 1, 2, \ldots, r+2)$ are constants independent of h and thus the proof is complete.

Proof of theorem 3.2.1.

By using equations (3.2.1) and (3.1.3) we can get

$$|y^{(q)}(x) - S_\Delta^{(q)}(x)| = \Bigg| \sum_{j=0}^{r-q} \frac{y_k^{(j+q)}}{j!}(x-x_k)^j + \frac{y^{(r+1)}(\xi_{kq})}{(r+1-q)!}(x-x_k)^{r+1-q} - $$
$$- \sum_{j=0}^{r-q} \frac{\bar{y}_k^{(j+q)}}{j!}(x-x_k)^j - \frac{\bar{y}_k^{(r+1)}}{(r+1-q)!}(x-x_k)^{r+1-q} -$$
$$- \sum_{p=1}^{r+2} q! \binom{p+r+1}{q} a_p^{(k)} (x-x_k)^{p+r+1-q} \Bigg|$$
$$\leqq \sum_{j=0}^{r-q+1} \frac{|y_k^{(j+q)} - \bar{y}_k^{(j+q)}|}{j!} h^j + \frac{|y^{(r+1)}(\xi_{kq}) - y_k^{(r+1)}|}{(r+1-q)!} h^{r+1-q} +$$
$$+ \sum_{p=1}^{r+2} q! \binom{p+r+1}{q} |a_p^{(k)}| \, h^{p+r+1-q}.$$

Using theorem 2.2.1, and 2.2.2 together with the definition of the modulus of continuity $\omega_r(h)$ and the lemma 3.2.1 respectively this will be

$$\leqq c_q^{**} \, \omega_r(h) \, h^{r+1-q}.$$

Taking $E = \max c_q^{**}$, where $q = 0, 1, \ldots, r+1$, we get

$$|y^{(q)}(x) - S_\Delta^{(q)}(x)| \leqq E \, \omega_r(h) \, h^{r+1-q}$$

and thus the proof of the theorem is complete.

At the end we shall estimate the error by which the constructed spline function, approximating the soiution, $S_\Delta(x)$ fails to satisfy the differential equation. Hence, we introduce the following theorem:

Theorem 3.2.2 If $\bar{S}'(x)$ denotes the function

$$\bar{S}'_\Delta(x) = f(x, S_\Delta(x)),$$

where $S_\Delta(x)$ is the spline function introduced in theorem 3.1, then for any $x \in [0, b]$ we have

$$|\bar{S}'_\Delta(x) - S'_\Delta(x)| \leqq M \, \omega_r(h) \, h^r,$$

where M is some constant independent of h.

Proof. We have

$$|\bar{S}'_\Delta(x) - S'_\Delta(x)| \leqq |\bar{S}'_\Delta(x) - y'(x)| + |y'(x) - S'_\Delta(x)|$$
$$= |f(x, S_\Delta(x)) - f(x, y(x))| + |y'(x) - S'_\Delta(x)|,$$

applying the Lipschitz condition on f this will be

$$\leqq L|S_\Delta(x)-y(x)|+|y'(x)-S'_\Delta(x)|,$$

applying theorem 3.2.1, this becomes

$$\leqq LE\ \omega_r(h)\,h^{r+1}+E\ \omega_r(h)\,h^r$$
$$\leqq M\ \omega_r(h)\,h^r,$$

where M is some constant independent of h, and thus the proof is complete.

Remark: In the case when $f\in C^\infty\,[0,\,b]$ we can choose r to be finite in such a way that the error will be in the allowable range, because as we have seen in the convergence theorems, the error is $0(h^{r+1})$. Also in practical applications if $f\in C^r\,[0,b]$, where r is large finite number, it is enough to choose a suitable smaller r in the sense that the error will be in the allowable range.

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