COMPUTATION OF THE EIGENSYSTEM OF SYMMETRIC FIVE DIAGONAL TOEPLITZ MATRICES

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(Received November 4, 1976)

1. Let

$$B_n = \begin{bmatrix} a_0 & a_1 & 1 & & & & \\ a_1 & a_0 & a_1 & 1 & & & \\ 1 & a_1 & a_0 & a_1 & 1 & & \\ & & \cdot & & & & \\ & & & & & & 1 \\ & & & & & \cdot & a_1 \\ & & & & 1 & a_1 & a_0 \end{bmatrix} \tag{1.1}$$

denote a five diagonal symmetric Toeplitz matrix with real elements. Our purpose is to give a rapid method for the determination of its eigenvalues, and eigenvectors.

The determination of the eigenvalues will be reduced to the computation of the roots of easily expressible polynomials. The components of the eigenvectors are simple functions of the corresponding eigenvalues. Our method is more simple than that can be achieved by a direct recursion formula for the characteristic polynomial of B_n.

It is obvious that, if B_n has the eigenvalues $\lambda_1, \dots, \lambda_n$, then the eigenvalues of $B_n + hI$ are $\lambda_1 + h, \dots, \lambda_n + h$, and the corresponding eigenvectors remain unchanged. Therefore we may translate a_0 as we want.
We choose

(1.2)
$$a_0 = \frac{a_1^2+8}{4}.$$

2. If λ is an eigenvalue and $\mathbf{x}^T = (x_1, \ldots, x_n)^T$ a corresponding eigenvector, then the following equations are satisfied:

(2.1)
$$\begin{cases} x_{i-2}+a_1 x_{i-1}+(a_0-\lambda)x_i+a_1 x_{i+1}+x_{i+2} = 0 \\ i = 1, 2, \ldots, n, \end{cases}$$

where

(2.2)
$$\begin{cases} x_0 = x_{-1} = 0, \\ x_{n+1} = x_{n+2} = 0. \end{cases}$$

Then we can extend the sequence $x_{-1}, x_0, \ldots, x_{n+1}, x_{n+2}$ so that (2.1) hold for every integer i.

Conversely, if the difference equation

(2.3)
$$\begin{cases} x_{i+2}+a_1 x_{i-1}+(a_0-\lambda)x_i+a_1 x_{i+1}+x_{i+2} = 0 \\ i = \ldots, -1, 0, 1, \ldots \end{cases}$$

has a non-trivial solution $\{x_i\}$ with the conditions (2.2), then λ is an eigenvalue and $(x_1, \ldots, x_n)^T$ a corresponding eigenvector of B_n.

3. Let

(3.1)
$$P(z) = 1+a_1 z+(a_0-\lambda)z^2+a_1 z^3+z^4$$

be the characteristic polynomial of (2.3). We have

(3.2)
$$\varphi(w) \doteq \frac{P(z)}{z^2} = w^2+a_1 w+(a_0-2-\lambda),$$

where
$$w = z+1/z.$$

Let W_1, W_2 be the roots of $\varphi(w)$. Observing (1.2) we have

(3.3)
$$W_1 = -\frac{a_1}{2}+\sqrt{\lambda}, \quad W_2 = -\frac{a_1}{2}-\sqrt{\lambda}.$$

Let $\Theta_1, \Theta^{-1}, \Theta_2, \Theta_2^{-1}$ be the roots of $P(z)$. Then

(3.4)
$$\Theta_i+\Theta_i^{-1} = W_i \quad (i = 1, 2)$$

hold. Let $\sigma_h (\doteq (\sigma_h(\lambda))$ denote the hth powersum of the roots of $P(z)$, i.e.

(3.5)
$$\sigma_h = \Theta_1^h+\Theta_1^{-h}+\Theta_2^h+\Theta_2^{-h}.$$

We have the recursion relation

(3.6)
$$\Theta_i^{h+1}+\Theta_i^{-(h+1)} = W_i[\Theta_i^h+\Theta_i^{-h}]-[\Theta_i^{h-1}+\Theta_i^{-(h-1)}].$$

Now we define the polynomials $g_j(K)$ by

$$
\begin{cases} g_{j+1}(K) = \left(-\dfrac{a_1}{2}+K\right) g_j(K) - g_{j-1}(K), \\ g_0(K) = 2, \; g_1(K) = -\dfrac{a_1}{2}+K. \end{cases} \tag{3.7}
$$

Comparing (3.6), (3.7) we have

$$
\sigma_h(\lambda) = g_h(\sqrt{\lambda}) + g_h(-\sqrt{\lambda}). \tag{3.8}
$$

4. Let J denote the reverse transformation in the n-dimensional space, i.e. transforming $(y_1, \ldots, y_n)^T$ to $(y_n, \ldots, y_1)^T$. From the symmetry of (2.1), (2.2) immediately follows that, if λ is an eigenvalue and $\mathbf{x}$ is a corresponding eigenvector, then $J\mathbf{x}$ is an eigenvector, too. Let L denote the subspace of the eigenvectors of B_n that correspond to a λ. We have that $JL \subseteq L$. It is obvious that the dimension of L is at most two, since the values x_1, x_2 (with $x_0 = x_{-1} = 0$) determine all the other components x_h in (2.3). We shall prove that for all eigenvalue λ corresponds only one eigenvector, apart from some special a_0. If the eigenspace L is of dimension one, and $\mathbf{x} \in L$, then $J\mathbf{x} = \alpha\mathbf{x}$, and from $J^2 = I$ it follows that $\alpha = 1$ or -1. We shall use this property for the investigation of the solution of (2.1).

5. Let us suppose that for the eigenvalue λ_0 there exist two independent eigenvector. Assume that the roots of $P(z, \lambda_0)$ are distinct. The general solution of (2.1), (2.2) has the form

$$
x_h = c_1 \Theta_1^h + c_3 \Theta_1^{-h} + c_2 \Theta_2^h + c_4 \Theta_2^{-h}, \tag{5.1}
$$

where c_1, c_2, c_3, c_4 are suitable constants. Since there are two eigenvectors, we have that the rank of the matrix

$$
\begin{vmatrix} \Theta_1^{-1} & \Theta_1 & \Theta_2^{-1} & \Theta_2 \\ 1 & 1 & 1 & 1 \\ \Theta_1^{n+1} & \Theta_1^{-(n+1)} & \Theta_2^{n+1} & \Theta_2^{-(n+1)} \\ \Theta_1^{n+2} & \Theta_1^{-(n+2)} & \Theta_2^{n+2} & \Theta_2^{-(n+2)} \end{vmatrix}
$$

is two. Let $\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3^T, \mathbf{r}_4^T$ denote the corresponding row vectors of it. It is obvious that $\mathbf{r}_1, \mathbf{r}_2$ are independent, and so

$$
\mathbf{r}_3 = A\mathbf{r}_1 + B\mathbf{r}_2, \tag{5.2}
$$

$$
\mathbf{r}_4 = C\mathbf{r}_1 + D\mathbf{r}_2, \tag{5.3}
$$

where A, B, C, D are suitable constants. From (5.2) we get that

$$
\Theta_i^{n+1} = A\Theta_i^{-1} + B, \quad \Theta_i^{-(n+1)} = A\Theta_i + B \quad (i = 1, 2), \tag{5.4}
$$

whence, by multiplying these equations,

$$1 = (A\Theta_i^{-1}+B)(A\Theta_i+B) = A^2+B^2+AB\,w_i \quad (i = 1, 2)$$

follows. Consequently (since $W_1 \neq W_2$)

$$A^2+B^2-1 = 0, \quad AB = 0,$$

that has the solutions $(A, B) = (0, -1), (0, +1), (-1, 0), (+1, 0)$. Substituting this into (5.4), we get the corresponding relations:

$$\Theta_i^{n+2} = -1 \quad (i = 1, 2); \qquad \Theta_i^{n+2} = +1 \quad (i = 1, 2);$$
$$\Theta_i^{n+1} = -1 \quad (i = 1, 2); \qquad \Theta_i^{n+1} = +1 \quad (i = 1, 2).$$

Similarly, (5.3) has the solutions $(C, D) = (0, -1), (0, +1), (-1, 0), (+1, 0)$ and the relations $\Theta_i^{n+3} = -1$, $\Theta_i^{n+3} = +1$, $\Theta_i^{n+2} = -1$, $\Theta_i^{n+2} = +1$ hold respectively. The equations (5.2), (5.3) hold simultaneously only if $\Theta_j^{n+2} = 1$ $(j = 1, 2)$, or $\Theta_j^{n+2} = -1$.

In the first case

$$\Theta_j = \exp\left(2\pi i \frac{k_j}{n+2}\right) \quad (j = 1, 2), \tag{5.5}$$

where k_1, k_2 are integers satisfying the conditions:

$$0 < k_j \leq n+1, \; k_1 \neq k_2, \; k_1+k_2 \neq n+2, \; k_j \neq \frac{n+2}{2} \quad (j = 1, 2). \tag{5.6}$$

On these conditions

$$\begin{cases} a_1 = -(w_1+w_2) = -2\left\{\cos 2\pi \dfrac{k_1}{n+2} + \cos 2\pi \dfrac{k_2}{n+2}\right\}, \\ \lambda = \left(\cos 2\pi \dfrac{k_1}{n+2} - \cos 2\pi \dfrac{k_2}{n+2}\right)^2. \end{cases} \tag{5.7}$$

In the second case

$$\Theta_j = \exp\left(2\pi i \frac{l_j}{2(n+2)}\right) \quad (j = 1, 2), \tag{5.8}$$

where l_1, l_2 are integers satisfying the conditions

$$\begin{cases} 0 < l_j \leq 2(n+2)-1, \; l_1 \neq l_2, \; l_j \neq n+2; \\ l_1+l_2 \neq 2(n+2), \\ l_1, l_2 \text{ odd}. \end{cases} \tag{5.9}$$

On these conditions

$$(5.10)\qquad \begin{cases} a_1 = -2\left\{\cos 2\pi \dfrac{l_1}{2(n+2)} + \cos 2\pi \dfrac{l_2}{2(n+2)}\right\}, \\ \lambda = \left(\cos 2\pi \dfrac{l_1}{2(n+2)} - \cos 2\pi \dfrac{l_2}{2(n+2)}\right)^2. \end{cases}$$

The determination of the corresponding eigenvectors is almost straighforward.

6. Let now suppose that λ_0 is an eigenvalue having only one eigenvector $\mathbf{x}$. Then $J\mathbf{x} = \alpha\mathbf{x}$ ($\alpha = 1$ or -1). Suppose that the roots of $P(z) = P(z, \lambda_0)$ are distinct. Then x_h has the form

$$(6.1)\qquad x_h = c_1 \Theta_1^h + c_3 \Theta_1^{-h} + c_2 \Theta_2^h + c_4 \Theta_2^{-h},$$

where c_1, c_2, c_3, c_4 are not identically vanishing suitable constants. Observing that $x_{n+1-h} = \alpha x_h$ for every h, we get

$$c_3 = \alpha c_1 \Theta_1^{n+1}, \quad c_4 = \alpha c_2 \Theta_2^{n+1}.$$

So we have

$$x_h = c_1 [\Theta_1^h + \alpha \Theta_1^{n+1-h}] + c_2 [\Theta_2^h + \alpha \Theta_2^{n+1-h}].$$

If $x_0 = x_1 = 0$, then $x_{n+1} = x_{n+2} = 0$. The conditions $x_0 = x_1 = 0$ hold if and only if the determinant

$$D_\alpha(\Theta_1, \Theta_2) \doteq \det \begin{vmatrix} \Theta_1^{-1} + \alpha \Theta_1^{n+2} & \Theta_2^{-1} + \alpha \Theta_2^{n+2} \\ 1 + \alpha \Theta_1^{n+1} & 1 + \alpha \Theta_2^{n+1} \end{vmatrix}$$

is zero.

Furthermore, taking $\Theta_1 \to \Theta_1^{-1}$, $\Theta_2 \to \Theta_2^{-1}$ we get that the conditions $x_0 = x_{-1} = 0$ hold if and only if

$$D_\alpha(\Theta_1^{-1}, \Theta_2^{-1}) = 0,$$

or if and only if

$$(6.2)\qquad R_\alpha(\lambda) \overset{\text{def}}{=} D_\alpha(\Theta_1, \Theta_2) \cdot D_\alpha(\Theta_1^{-1}, \Theta_2^{-1})$$

is zero at $\lambda = \lambda_0$.

So we have proved the following assertion. If for a λ_0 the roots of $P(z, \lambda_0)$ are distinct and the corresponding eigenspace is one-dimensional, then λ_0 is an eigenvalue of B_n if and only if $R_\alpha(\lambda_0) = 0$ for $\alpha = 1$ or -1.

By an easy calculation (multiplying the determinants by the row by row rule) we get:

$$(6.3)\qquad R_\alpha(\lambda) = (4 + \alpha \sigma_{n+3})(4 + \alpha \sigma_{n+1}) - (\sigma_1 + \alpha \sigma_{n+2})^2.$$

7. Now we discuss the cases, when $P(z, \lambda)$ has a multiple root.

A) $W_1 = W_2 = 2$.

In this case $a_1 = 0$, $a_0 = 2$, $\lambda = 4$ and the corresponding difference equation has the form:

$$x_{r+2} - 2x_r + x_{r+2} = 0 \quad (r = 1, \ldots, n)$$

$$x_0 = x_{-1} = 0, \quad x_{n+1} = x_{n+2} = 0.$$

Considering this separately for odd and even indices, we see immediately that this has no non-trivial solution.

B) $W_1 = W_2 = 2$ or -2.

In this case $\lambda = 0$, $a_0 = 6$ and $a_1 = \mp 4$. Then $P(z, \lambda)$ has the root $\Theta = \pm 1$ with multiplicity 4, the general solution of the corresponding difference equation is

$$x_h = [c_1 + c_2 h + c_3 h^2 + c_4 h^3]\,\Theta^h. \tag{7.1}$$

To the fulfilment of $x_{-1} = x_0 = x_{n+1} = x_{n+2} = 0$ it needs that the third-degree polynomial on the right hand side of (7.1) vanished at $h = 0, -1, n+1, n+2$. Consequently x_h vanishes identically, there is no eigenvalue.

C) $W_1 = W_2$ $(\neq \pm 2)$.

Now the general solution of (2.3) has the form

$$x_h = (c_1 + c_3 h)\,\Theta^h + (c_2 + c_4 h)\,\Theta^{-h},$$

$$\Theta + 1/\Theta = W_i.$$

There is an eigenvector if and only if the determinant

$$D(\Theta) = \text{def} \begin{vmatrix} \Theta^{-1} & -\Theta^{-1} & \Theta & -\Theta \\ 1 & 0 & 1 & 0 \\ \Theta^{n+1} & (n+1)\,\Theta^{n+1} & \Theta^{-(n+1)} & (n+1)\,\Theta^{-(n+1)} \\ \Theta^{n+2} & (n+2)\,\Theta^{n+2} & \Theta^{-(n+2)} & (n+2)\,\Theta^{-(n+2)} \end{vmatrix}$$

is zero. After an easy computation we get

$$D(\Theta) = \Theta^{2n+4} + \Theta^{-(2n+4)} - (n+2)^2\,[\Theta^2 + \Theta^{-2}] + 2\,(n+1)\,(n+3).$$

Since in our case $W_i = -\dfrac{a_1}{2}$ is real, therefore Θ must be real or unimodular.

Let Θ be real. Observing that $D(\Theta) = D(-\Theta) = D(1/\Theta)$, we may assume that $\Theta > 1$. By the substitution $\Theta^2 = e^\tau\,(\tau \geq 0)$ we have

$$D(\Theta) = l\big((n+2)\,\tau\big) - (n+2)^2\,l(\tau),$$

where

$$l(\tau) = e^\tau + e^{-\tau} - 2.$$

Considering the power series expansion at $\tau = 0$ of $l(\tau)$, we get the inequality

$$l((n+2)\tau) > (n+2)^2 l(\tau) \quad (\tau > 0).$$

So all the real solutions of $D(\Theta) = 0$ are $\Theta = \pm 1$. Let now $\Theta = e^{i\varphi}$ (φ real). We have

$$D(e^{i\varphi}) = 2(n+2)^2[1-\cos 2\varphi] - 2[1-\cos(2n+4)\varphi]$$
$$= 4(n+2)^2 \sin^2\varphi - 4\sin^2(n+2)\varphi,$$

and so

$$D(e^{i\varphi}) > 0$$

for $\varphi \neq 0$.

There is no eigenvalue.

D) $W_1 = 2$, $W_2 \neq \pm 2$.

In this case

$$W_2 = -a_1 - W_1 = -a_1 - 2, \; a_0 - \lambda = -2(1+a_1), \; \lambda = 4 + 2a_1 + \frac{a_1^2}{4}.$$

Now the general solution of (2.3) has the form

$$x_h = c_1 + c_3 h + c_2 \Theta^h + c_4 \Theta^{-h} \quad (\Theta + 1/\Theta = W_2). \tag{7.2}$$

Suppose that there are two independent eigenvectors. Then the rank of th matrix

$$\begin{vmatrix} 1 & -1 & \Theta^{-1} & \Theta \\ 1 & 0 & 1 & 1 \\ 1 & n+1 & \Theta^{n+1} & \Theta^{-(n+1)} \\ 1 & n+2 & \Theta^{n+2} & \Theta^{-(n+2)} \end{vmatrix}$$

is two. Let $\vartheta_1^T, \vartheta_2^T, \vartheta_3^T, \vartheta_4^T$ denote the row vectors of it. It is obvious that ϑ_1, ϑ_2 are independent. Suppose that

$$\vartheta_3 = A\vartheta_1 + B\vartheta_2.$$

Considering the first two components, we get $A = -(n+1)$, $B = (n+2)$. Furthermore, by substituting the last two components, we have

$$\Theta^{n+1} = A\Theta^{-1} + B, \; \Theta^{-(n+1)} = A\Theta + B,$$

whence, by multiplying them, $1 = A^2 + B^2 + ABW_2$. Observing that

$$A^2 + B^2 - 1 = 2(n+1)(n+2), \quad AB = -(n+1)(n+2),$$

we get $W_2 = 2$. This case was above considered. Now we have that

$$x_{n+1-h} = \alpha x_h \quad (\alpha = 1 \text{ or } -1).$$

We get

$$0 = x_{n+1-h} - \alpha x_h = [(1-\alpha)c_1 + (n+1)c_3] + [-c_3 - \alpha c_3]h + \\ + [c_4 \Theta^{-(n+1)} - \alpha c_2]\Theta^h + [c_2 \Theta^{n+1} - \alpha c_4]\Theta^{-h}.$$

First we consider the symmetric case: $\alpha = 1$. Then $c_3 = 0$, $c_4 = c_2 \Theta^{n+1}$ and

$$0 = x_{-1} c_1 + c_2 [\Theta^{-1} + \Theta^{n+2}],$$
$$0 = x_0 = c_1 + c_2 [1 + \Theta^{n+1}].$$

This has a non-trivial solution if and only if $\Theta^{-1} + \Theta^{n+2} = 1 + \Theta^{n+1}$, i.e if $\Theta^{n+2} = 1$. In this case $W_2 = 2\cos\frac{2k\pi}{n+2}$ $(k = 0, \ldots, n)$. We must give up $k = 0$, and $k = \frac{n+2}{2}$ for even n. In the other cases of k, λ is an eigenvalue:

$$\lambda = 4 + 2a_1 + \frac{a_1^2}{4}, \quad a_1 = -2\left[1 + \cos\frac{2k\pi}{n+2}\right].$$

Let now consider the unsymmetric case: $\alpha = -1$. Then $2c_1 + c_3(n+1) = 0$, $c_4 = -c_2 \Theta^{n+1}$, and from $x_{-1} = 0$, $x_0 = 0$ we get that the condition

$$u(\Theta) = \det \begin{vmatrix} n+3 & \Theta^{n+2} - \Theta^{-1} \\ n+1 & \Theta^{n+1} - 1 \end{vmatrix} = 0$$

holds, if λ is an eigenvalue. By a simple computation we can prove that $u(\Theta) \neq 0$ if Θ is real and $\Theta \neq 1, -1$. Taking $\Theta = e^{i\varphi}(0 < \varphi < \pi)$ we get

$$u(\Theta) = 2i\,\Theta^{\frac{n+1}{2}} \left[(n+3)\sin(n+1)\frac{\varphi}{2} - (n+1)\sin(n+3)\frac{\varphi}{2}\right].$$

Let

$$L(\eta) \doteq (n+3)\sin(n+1)\eta - (n+1)\sin(n+3)\eta.$$

It is easy to see that in the interval $\left(0, \frac{\pi}{2}\right)$ $L(\eta)$ has k distinct roots, where

$$k_n = 2\left[\frac{n+2}{2}\right] + l_n,$$

$$l_n = \begin{cases} -1 & \text{if } n+2 \equiv 0 \pmod 4 \\ 0 & \text{if } n+2 \equiv 1 \pmod 4 \\ 0 & \text{if } n+2 \equiv 2 \pmod 4 \\ 1 & \text{if } n+2 \equiv 3 \pmod 4 \end{cases}.$$

Let $\eta_1 < \ldots < \eta_{k_n}$ denote these roots. Then $u(e^{i \cdot 2\eta_j}) = 0$ $(j = 1, \ldots, k_n)$, and the corresponding λ is an eigenvalue.

E) $W_1 = -2$, $W_2 \neq \pm 2$.

There is a connection with the case *D)*. Suppose that for some $(a_1 =) a_1^*$ we get an eigenvector $\mathbf{y} = (y_1, \ldots, y_n)^T$. Then

$$W_2 = 2 - a_1^*, \quad a_0^* - \lambda = -2[1 - a_1^*]$$

and

$$y_{j-2} + a_1^* y_{j-1} - 2(1 - a_1^*) y_j + a_1^* y_{j+1} + y_{j+2} = 0 \quad (j = 1, \ldots, n)$$

$$y_{-1} = y_0 = y_{n+1} = y_{n+2} = 0\,.$$

If we take $x_j = (-1)^j y_j$, then

$$x_{j-2} + (-a_1^*) x_{j-1} - 2[1 + (-a_1^*)] x_j + (-a_1^*) x_{j+1} + x_{j+2} = 0\,.$$

Thus for taking $a_1 = -a_1^*$, we get the case *D)*.