COMPUTATION OF THE EIGENSYSTEM OF TOEPLITZ BAND MATRICES

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1. *Introduction.* In this paper we shall outline a method for the computation of the eigenvalues of band Toeplitz matrices. By this method we can determine their eigenvectors as well. In a forthcoming paper we shall report on the stability of the method and some numerical experiments concerning it. Previously we have worked out a method for the symmetric five diagonal case.

2. *Reduction of the problem for finding the roots of a polynomial.* Let

$$
(2.1) \qquad B_n = \begin{bmatrix}
\Phi_0 & \cdots & \Phi_{-p} & & & \\
\vdots & \ddots & & \ddots & & \\
\Phi_q & & \ddots & & \ddots & \Phi_{-p} \\
 & \ddots & & \ddots & & \vdots \\
 & & \ddots & & \ddots & \\
 & & & \Phi_q & \cdots & \Phi_0
\end{bmatrix}
$$

be a band Toeplitz matrix of order n. We assume that its elements are complex numbers.

It is obvious that λ is an eigenvalue of B_n, if and only if the difference equation

$$
\begin{aligned}
&\Phi_q x_{i-q} + \dots + \Phi_1 x_{i-1} + (\Phi_0 - \lambda) x_i + \\
(2.2) \qquad &+ \Phi_{-1} x_{i+1} + \dots + \Phi_{-p} x_{i+p} = 0 \quad i = 1, \dots, n
\end{aligned}
$$

with the conditions

$$\begin{cases} x_0 = \ldots = x_{1-q} = 0 \\ x_{n+1} = \ldots = x_{n+p} = 0 \end{cases} \tag{2.3}$$

has a non-trivial solution. Then the solution vector $(x_1, \ldots, x_n)^T$ is a corresponding eigenvector. A solution of (2.2) satisfying (2.3) can be extended so that

$$\begin{aligned} &\Phi_q x_{i-q} + \ldots + \Phi_1 x_{i-1} + (\Phi_0 - \lambda) x_i + \\ &+ \Phi_{-1} x_{i+1} + \ldots + \Phi_{-p} x_{i+p} = 0 \end{aligned} \tag{2.4}$$

hold for every integer i.

We know the form of the general solution of (2.4).

Let

$$P(z; \lambda) = P(z) = \sum_{j=-q}^{p} \tilde{\Phi}_{-j} z^{j+q}, \tag{2.5}$$

$$\tilde{\Phi}_k = \begin{cases} \Phi_k & \text{if } k \neq 0 \\ \Phi_0 - \lambda & \text{if } k = 0 \end{cases} \tag{2.6}$$

be the characteristic polynomial of (2.4).

Suppose that λ_0 is an eigenvalue of B_n. Let $w_1, w_2, \ldots, w_l$ be all the distinct roots of $P(z; \lambda_0)$ with the corresponding $k_1, k_2, \ldots, k_l$ multiplicities. Then the general solution of (2.4) can be written in the form

$$x_h = \sum_{j=1}^{l} Q_j(h) \cdot w_j^h, \tag{2.7}$$

where

$$Q_j(h) = \sum_{i=0}^{k_j-1} c_{i,j} \cdot h^i \tag{2.8}$$

is a polynomial of degree $k_j - 1$.

Let R denote the set $\{1-q, \ldots, 0, n+1, \ldots, n+p\}$, let $r = p+q$, and the r-dimensional vectors $\mathbf{y}_h$, $\mathbf{c}$ be defined by

$$\mathbf{y}_h^T = [W_1^h, h W_1^h, \ldots, h^{k_1-1} W_1^h, \ldots, W_l^h, h W_l^h, \ldots, h^{k_l-1} W_l^h], \tag{2.9}$$

$$\mathbf{c}^T = [c_{0,1}, \ldots, c_{k_1-1,1}, \ldots, c_{0,l}, \ldots, c_{k_l-1,1}]. \tag{2.10}$$

By this notation we can write

$$x_h = (\mathbf{y}_h, \mathbf{c}),$$

where (,) denotes the scalar product. Thus the relations (2.3) can be written as

$$(y_h, c) = 0 \quad h \in R. \tag{2.11}$$

(2.11) has a non-trivial solution $\mathbf{c}$, if and only if the dimension of the space spanned by the vectorials $\mathbf{y}_h$ $(h\in R)$ is smaller than r, i.e. if the Gram-determinant of $\mathbf{y}_h$ $(h\in R)$ is zero:

$$G(\{\mathbf{y}_h\}) \overset{\text{def}}{=} \det |(\mathbf{y}_h, \mathbf{y}_k)|_{h,k\in R} = 0. \tag{2.12}$$

For the computation of the Gram determinant (allowing the existence of multiple roots) the knowing of the roots $W_1, \ldots, W_l$ is needed. Assuming that the roots of $P(z, \lambda_0)$ are simple, we get the more convenient form:

$$\mathbf{y}_h^T = [w_1, \ldots, w_r], \tag{2.13}$$

$$\mathbf{c}^T = [c_1, \ldots, c_r]. \tag{2.14}$$

Furthermore we have that

$$(\mathbf{y}_h, \mathbf{y}_k) = \sigma_{h+k}, \tag{2.15}$$

where

$$\sigma_l = \sum_{j=1}^{r} w_j^l. \tag{2.16}$$

So we get

$$G(\{\mathbf{y}_h\}) = \det |\sigma_{h+k}|_{h,k\in R}. \tag{2.17}$$

In what follows, for a general λ let $R(\lambda)$ be the determinant

$$\det |\sigma_{h+k}|_{h,k\in R} \left(\overset{\text{def}}{=} R(\lambda)\right), \tag{2.18}$$

where $\sigma_j = \sigma_j(\lambda)$ denotes the j th powersum of the roots of $P(z, \lambda)$.

It is obvious that $R(\lambda)$ is a polynomial of λ. For the computation of its zeros we can use any of the root-finding methods.

It is well known that if $P(z, \lambda)$ has a multiple root, then the discriminant $D(\lambda)$ of it must be zero. $D(\lambda)$ is a polynomial of degree $2(p+q)-1$.

So we can go on the following way. First compute the roots of $D(\lambda)$, and for every root λ determine the roots of $P(z, \lambda)$. After then, by computing (2.12) we decide that whether λ is an eigenvalue.

The other eigenvalues of B_n must satisfy the relation $R(\lambda) = 0$.

3. *Recursion formulas for* σ_l.

If we want to compute the roots of $R(\lambda)$, we must to compute $\sigma_j = \sigma_j(\lambda)$ for many λ and for indices $j = h+k$, $h, k\in R$. To take this easy, we can use the Newton–Girard formulas.

We consider the polynomial

$$Q(z) \overset{\text{def}}{=} \frac{P(z, \lambda)}{\Phi_{-p}} = \sum_{j=-q}^{p} \frac{\tilde{\Phi}_j}{\Phi_{-p}} z^{j+q} = \sum_{k=0}^{r} s_k z^{r+k},$$

where

$$s_k = \frac{\tilde{\Phi}_{r+k+q}}{\Phi_{-p}}.$$

By taking

$$\begin{cases} a_k = (-1)^k s_k & (k = 1, \ldots, r) \\ a_0 = 1 \end{cases}, \tag{3.1}$$

the Newton – Girard formulas give that

$$\begin{cases} \sigma_j + a_1 \sigma_{j-1} + \ldots + a_{j-1} \sigma_1 + j\, a_j = 0 & (j = 1, \ldots, r) \\ \sigma_{r+j} + a_1 \sigma_{r+j-1} + \ldots + a_r \sigma_j = 0 & (j = 1, 2, \ldots \end{cases}. \tag{3.2}$$

We can deduce similar formulas for σ_{-j}. Since

$$Q(z) = \prod_{j=1}^{r} (z - w_l),$$

therefore

$$z^r Q(1/z) = a_r \prod_{j=1}^{r} (z - 1/w_j).$$

Since $\sigma_{-j} = \sum (1/w_l)^j$, therefore by

$$a \quad = (-1)_k \frac{s_{r-k}}{a_r}$$

we get

$$\begin{aligned} &\sigma_{-j} + a_1^* \sigma_{-(j-1)} + \ldots + a_{j-1}^* \sigma_{-1} + j\, a_j^* = 0 \quad (j = 1, 2, \ldots, r) \\ &\sigma_{-(r+j)} + a_1^* \sigma_{-(r+j-1)} + \ldots + a_r^* \sigma_{-j} = 0 \quad (j = 1, 2, \ldots). \end{aligned} \tag{3.3}$$

If we use Newton – Raphson method for searching the roots of $R(\lambda)$, we must compute the derivative of $\sigma_j = \sigma_j(\lambda)$, too. We have

$$a_\nu = \begin{cases} (-1)^\nu \dfrac{\Phi_{\nu+q-r}}{\Phi_{-p}}, & \text{if} \quad \nu \neq p, \\[2ex] (-1)^p \dfrac{\Phi_{0-\lambda}}{\Phi_{-p}}, & \text{if} \quad \nu = p. \end{cases}$$

$$a_\nu^* = \begin{cases} (-1)^{r-\nu} \dfrac{\Phi_{-\nu+q}}{\Phi_q}, & \text{if} \quad \nu \neq q, \\[2ex] (-1^p \dfrac{\Phi_{0-\lambda}}{\Phi_{-p}}, & \text{if} \quad \nu = q. \end{cases}$$

We see that only the coefficients a_p, a_q^* depend on λ, and

$$\frac{da_p(\lambda)}{d\lambda} = \frac{(-1)^{p+1}}{\Phi_{-p}}, \quad \frac{da_q^*(\lambda)}{d\lambda} = \frac{(-1)^{p+1}}{\Phi_q}.$$

So, by differenciating the Newton – Girard formulas, we get

$$\sigma'_j + a_1 \sigma'_{j-1} + \ldots + a_{j-1} \sigma'_1 = 0 \quad (j = 1, 2, \ldots, p-1)$$

$$\sigma'_p + a_1 \sigma'_{p-1} + \ldots + a_{p-1} \sigma'_1 + \frac{(-1)^{p+1} p}{\Phi_{-p}} = 0$$

$$\sigma'_j + a_1 \sigma'_{j-1} + \ldots + a_{j-1} \sigma'_1 + \frac{(-1)^{p+1}}{\Phi_{-p}} \sigma_{j-p} = 0 \quad (j = p+1, \ldots, r),$$

$$\sigma'_{r+j} + a_1 \sigma'_{r+j-1} + \ldots + a_r \sigma'_j + \frac{(-1)^{p+1}}{\Phi_{-p}} \sigma_{q+j} = 0 \quad (j = 1, 2, \ldots).$$

Observing that $\sigma'_1 = \ldots = \sigma'_{p-1} = 0$, we can write the previous recursion formula in the form

$$\sigma'_j = 0 \quad (j = 1, 2, \ldots, p-1)$$

$$\sigma'_p = \frac{(-1)^p p}{\Phi_{-p}}$$

$$\sigma'_j + a_1 \sigma'_{j-1} + \ldots + a_{j-p} \sigma'_p + \frac{(-1)^{p+1}}{\Phi_{-p}} \sigma_{j-p} = 0 \quad (j = p+1, \ldots, r)$$

$$\sigma'_{r+j} + a_1 \sigma'_{r+j-1} + \ldots + a_r \sigma'_j + \frac{(-1)^{p+1}}{\Phi_{-p}} \sigma_{q+j} = 0 \quad (j = 1, 2, \ldots).$$

We can deduce similar formula for the derivatives of σ_j with negative indices.